Dynamic Programming and Recursive Representation

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RIEF

Introduction

- We want to study problems of the following form:

$$sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \text{(SP)}$$

$$s.t.$$

$$x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, ...$$

$$x_0 \in X \quad \text{given}$$

Corresponding to any such problem, we have a functional equation of the form:

$$v(x) = sup_{y \in \Gamma(x)} \left[F(x, y) + \beta v(y) \right], \quad \forall x \in X$$

The Principle of Optimality

- The Principle of Optimality allow us to study the relationship between solutions to the problems (SP) and (fE).
- Define A as the graph of Γ:

$$A = \{(x, y) \in XxX : y \in \Gamma(x)\}$$

  And let the real-valued function $F : A \to \mathbb{R}$ be the one-period return function, and let $\beta \geq 0$ the stationary discount factor. Thus the 'givens' for the problem are X, Γ, F and β.

The Principle of Optimality

- Call any sequence $\{x_t\}_{t=0}^{\infty}$ in X a plan. Given $x_0 \in X$, let:

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t), t = 0, 1, ...\}$$

  is the set of plans that are *feasible* from x_0. This means, $\Pi(x_0)$ is the set of sequences that satisfy the constraints in (SP).

- Now let's impose some assumptions to guarantee both the equivalence between the FE and the SP.

- **Assumption 4.1:** $\Gamma(x)$ is nonempty, forall $x \in X$.
- **Assumption 4.2:** For all $x_0 \in X$ and $\hat{x} \in \Pi(x_0)$, $\text{lím}_{n\rightarrow\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists (although it may be plus or minus infinity).

The Principle of Optimality - A.4.1 and A.4.2

Theorem

Let X, Γ, F, and β satisfy Assumptions 4.1-4.2. Then the function v^* satisfies (FE).

The Principle of Optimality - A.4.1 and A.4.2

Theorem

Let X, Γ, F, and β satisfy Assumptions 4.1-4.2. If v is a solution to (FE) and satisfies:

$$\lim_{n\to\infty} \beta^n v(x_n) = 0, \quad all \quad (x_0, x_1, ...) \in \Pi(x_0), all \quad x_0 \in X$$

Then $v = v^*$

The Principle of Optimality - A.4.1 and A.4.2

Theorem

Let X*,* Γ*,* F*, and* β *satisfy Assumptions 4.1-4.2 and let* $\hat{x}^* \in \Pi(x_0)$ *be a feasible plan that attains the supremum in (SP) for initial state* x_0*. Then:*

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*), \quad t = 0, 1, 2, ...$$

Then $v = v^*$

Bounded Returns - Existence

Now we study some assumptions that are needed to guarantee both the existence and the uniqueness of a solution in our problem of interest.

Assumption 4.3: X is a convex subset of $\mathbb{R}^l$, and the correspondence $\Gamma : X \to X$ is nonempty, compact-valued, and continuous.

Assumption 4.4: The function $F : A \to \mathbb{R}$ is bounded and continuous, and $0 < \beta < 1$.

Bounded Returns - Existence

Theorem

Let X, Γ, F, and β satisfy Assumptions 4.3-4.4 and let $C(X)$ be the space of bounded and continuous functions $f : X \to \mathbb{R}$, with the sup norm. Then the operator T maps $C(X)$ into itself, $T : C(X) \to C(X)$, T has a unique fixed point $v \in C(X)$; and for all $v_0 \in C(X)$,

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v||, \quad n = 0, 1, 2, ...$$

Moreover, given v, the optimal policy correspondence $G : X \to X$ defined by (2) is compact-valued and uhc.

Bounded Returns - Monotonicity

Now we describe some assumptions that whenever they are present, we have that the value function inherits some characteristics of the return function.

Assumption 4.5: For each y, F(.,y) is strictly increasing in each of its l arguments.

Assumption 4.6: Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$.

Bounded Returns - Monotonicity

Theorem

Let X, Γ, F, and β satisfy Assumptions 4.3-4.6, and let v be the unique solution to the SP. Then v is strictly increasing.

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v||, \quad n = 0, 1, 2, ...$$

Moreover, given v, the optimal policy correspondence $G : X \to X$ defined by (2) is compact-valued and uhc.

Bounded Returns - Concavity

- **Assumption 4.7:** F is strictly concave, that is:

$$F\left[\theta(x,y)] + (1-\theta)(x',y')\right] > \theta F(x,y) + (1-\theta)F(x',y')$$

all (x,y), (x',y')$\in A$, and all $\theta \in (0,1)$.

- **Assumption 4.8:** Γ is convex in the sense that for any $0 \leq \theta \leq 1$, and $x, x' \in X$,

$$y \in \Gamma(x) \quad \text{and} \quad y' \in \Gamma(x') implies$$
$$\theta y + (1-\theta)y' \in \Gamma\left[\theta x + (1-\theta)x'\right]$$

Bounded Returns - Concavity

Theorem

Let X, Γ, F, and β satisfy Assumptions 4.3-4.4 and 4.7-4.8, and let v be the unique solution to the SP, and let G be the policy rules. Then v is strictly concave and G is a continuous, single-valued function.

Bounded Returns - Differentiability

- **Assumption 4.9:** F is continuously differentiable on the interior of A.

Bounded Returns - Differentiability

Theorem

Let X, Γ, F, and β satisfy Assumptions 4.3-4.4 and 4.7-4.9, and let v be the unique solution to the SP, and let G be the policy rules. If $x_0 \in int(X)$ and $g(x_0) \in \Gamma(x_0)$, then v is continuously differentiable at x_0, with derivatives given by:

$$v_i(x_0) = F_i[x_0, g(x_0)], \quad i = 1, 2, ..., l.$$