Introduction to Real Analysis

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Introduction

- In this course we are going to learn new methods to solve general equilibrium models.
- These methods will allow us to solve more complex model in a compact and 'easy' way.
- However, to understand these methods we need some mathematical definitions and tools.
- The purpose of this class is showing you some of these notions and tools. Read Chapter 3 of SLP for more details.

Basic Definitions

- **Metric Space:** It is a set $\mathbb{S}$ together with a metric (a notion of distance) $\rho : \mathbb{S} x \mathbb{S} \to \mathbb{R}$, such that $\forall\ x, y, z \in \mathbb{S}$:

  1. $\rho(x, y) \geq 0$, with equality iff $x = y$
  2. $\rho(x, y) = \rho(y, x)$ y
  3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Basic Definitions

- **Normed Vector Space:** It is a vector space $\mathbb{S}$, together with a norm $\|.\| : \mathbb{S} \to \mathbb{R}$, such that $\forall\ x, y \in S$ y $\alpha \in \mathbb{R}$:
    1. $\|x\| \geq 0$, with equality iff $x = 0$
    2. $\|\alpha x\| = |\alpha|\|x\|$, y
    3. $\|x + y\| \leq \|x\| + \|y\|$

Basic Definitions

Definition: A sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbb{S}$ **converges** to $x \in \mathbb{S}$, if:

$$\forall \epsilon > 0, \exists N_\epsilon \quad \text{such that:} \quad \rho(x_n, x) < \epsilon, \quad \forall n \geq N_\epsilon$$

Definition: A sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbb{S}$ is a **Cauchy's sequence** if:

$$\forall \epsilon > 0, \exists N_\epsilon \quad \text{such that:} \quad \rho(x_n, x_m) < \epsilon, \forall n, m \geq N_\epsilon$$

Definition: A metric space $(\mathbb{S}, \rho)$ is **complete** if every Cauchy's sequence in $\mathbb{S}$ converges to an element in $\mathbb{S}$.

An example of this is the set of real numbers with $\rho(x, y) = |x - y|$

Theorem 1

Theorem

Let $X \subseteq \mathbb{R}$, and let $C(X)$ be the set of continuous and bounded functions $f : X \to \mathbb{R}$ with the sup norm, $\|f\| = sup_{x \in X}|f(x)|$. Then $C(X)$ is a complete norm vector space.

The Contraction Mapping Theorem

Definition: Let $(\mathbb{S}, \rho)$ a metric space and $T : \mathbb{S} \rightarrow S$ a function that maps S with itself. We say that T is a **contraction** with modulus β if for some $\beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$, $\forall x, y \in \mathbb{S}$.

The Contraction Mapping Theorem

Theorem

If $(\mathbb{S}, \rho)$ *is a complete metric space and* $T : \mathbb{S} \to \mathbb{S}$ *is a contraction with modulus* β*, then:*

1. T *has exactly a fixed point* v *in* $\mathbb{S}$*, and*
2. *for any* $v_0 \in \mathbb{S}$*,* $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$*,* $n = 0, 1, 2, ...$

The Contraction Mapping Theorem - Corollary 1

Theorem

Let $(\mathbb{S}, \rho)$ be a complete metric space and $T : \mathbb{S} \to \mathbb{S}$ a contraction with a fixed point $v \in \mathbb{S}$. If $\mathbb{S}'$ is a closed subset of $\mathbb{S}$ and $T(\mathbb{S}') \subseteq \mathbb{S}'$, then $v \in \mathbb{S}'$. If additionally $T(\mathbb{S}') \subseteq \mathbb{S}''$, then we have that $v = Tv \in \mathbb{S}''$

The Contraction Mapping Theorem - Corollary 2

Theorem

Let $(\mathbb{S}, \rho)$ *be a complete metric space and* $T : \mathbb{S} \to \mathbb{S}$*. Suppose that for some integer* N*,* $T^N : \mathbb{S} \to \mathbb{S}$ *is a contraction with modulus* β*. Then:*

- *T has exactly a fixed point in S, and*
- *for any* $v_0 \in \mathbb{S}$*,* $\rho(T^{kN} v_0, v) \leq \beta^k \rho(v_0, v)$*,* $k = 0, 1, 2, \ldots$

Blackwell's sufficient conditions

Theorem

Let $X \subseteq \mathbb{R}^l$, and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying:

- *(monotonicity)* $f, g \in B(X)$ *and* $f(x) \leq g(x)$, $\forall x \in X$, *implies* $(Tf)(x) \leq (Tg)(x), \forall x \in X$;
- *(discounting)* $\exists$ *some* $\beta \in (0,1)$ *such that:*

$$[T(f+a)](x) \leq (Tf)(x) + \beta a, \forall f \in B(X), a \geq 0, x \in X$$

Here $(f+a)(x)$ is the function defined by $(f+a)(x) = f(x) + a$. Then T is a contraction with modulus β.

The Theorem of the Maximum - Correspondences

- Our final goal is to analyze dynamic programming problems. Let x be the beginning-of-period state variable, an element $X \subseteq \mathbb{R}^l$, and $y \in X$ is the end-of-period state to be chosen.
- We will describe a correspondence from a set X into a set Y as a relation that assigns a set $\Gamma(x) \subseteq Y$ to each $x \subseteq X$ (we will be interested in the case $Y = X$).

The Theorem of the Maximum - Correspondences

Definition: A correspondence $\Gamma : X \rightarrow Y$ is lower hemi-continuous (lhc) at x if $\Gamma(x)$ is nonempty and if, for every $y \in \Gamma(x)$ and every sequence $x_n \rightarrow x$, there exists $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \rightarrow y$ and $y_n \in \Gamma(x_n)$, $\forall n \geq N$.

Definition: A correspondence $\Gamma : X \rightarrow Y$ is upper hemi-continuous (uhc) at x if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \rightarrow x$ and every sequence $\{y_n\}$ such that $y_n \in \Gamma(x_n)$, all n, there exists a convergent subsequence of y_n whose limit point y is in $\Gamma(x)$.

Definition: A correspondence : $X \rightarrow Y$ is **continuous** at $x \in X$ if it is both uhc and lhc at x.

The Theorem of the Maximum

Theorem

Let $X \subseteq \mathbb{R}^l$, *and* $Y \subseteq \mathbb{R}^m$, *let* $f : XxY \to \mathbb{R}$ *be a continuous function, and let:* $\Gamma : X \to Y$ *be a compact-valued and continuous correspondence. Then the function* $h : X \to \mathbb{R}$ *is continuous and the correspondence* $G : X \to Y$ *defined in (2) is nonempty, compact-valued, and uhc.*

Here $(f + a)(x)$ is the function defined by $(f + a)(x) = f(x) + a$. Then T is a contraction with modulus β.