Neoclassical Growth Model

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RIEF

Introduction

- General Equilibrium model with two types of agents:
  1. Large number of infinitely lived identical households (representative household).
  2. Large number of identical firms that produce a single good (representative firm).

Households

- Representative household of size L_t:
  - Preferences: intertemporal utility function:

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t/L_t)$$

u satisfies $u' > 0$ y $u'' < 0$ y

$$\text{lím}_{c \to 0} u'(c) = \infty$$

  - Endowments: Labor and capital (rented to firms).

Budget Constraint

- Households face the following budget constraint:

$$C_t + I_t = w_t L_t + r_t K_t + \Pi_t$$

  Price of consumption good normalized to 1, $\forall t$.

- Capital follows a law of motion:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

- Number of workers grow at a rate η:

$$L_{t+1} = (1 + \eta)L_t$$

Technology

- Aggregate production function.

$$Y_t = F(K_t, L_t)$$

  Technology satisfies (i) constant returns to scale, (ii) concavity y (iii) Inada conditions.

- Objective function: Profits:

$$\Pi_t = Y_t - w_t L_t - r_t K_t$$

Model in intensive units

- (variables c_t, i_t, K_t, y_t expressed in units of the unique good per worker)

$$u\left(\frac{C_t}{L_t}\right) = u(c_t)$$

$$\frac{K_{t+1}}{L_t} = (1+\eta)k_{t+1}$$

$$y_t = \frac{Y_t}{L_t} = F(\frac{K_t}{L_t},1) = f(k_t)$$

with $f' > 0$ y $f'' > 0$ y

$$\lim_{k \to 0} f'(k) = \infty \quad \lim_{k \to \infty} f'(k) = 0$$

Competitive Equilibrium

- A competitive equilibrium is a set of sequences for the sequences c_t, i_t, y_t y k_{t+1} and prices $w_t y r_t$ such that:
  - Given $k_0 > 0$. w_t y r_t, the quantities $\{c_t.\ i_t \text{ y } k_{t+1}\}_{t=0}^{\infty}$ solve the problem of the representative household:

$$max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.a.$$

$$c_t + i_t = w_t + r_t k_t \quad \forall t$$

$$(1+\eta)k_{t+1} = (1-\delta)k_t + i_t \quad \forall t$$

$$c_t, i_t \geq 0$$

Competitive Equilibrium

- In each period t, given w_t and r_t, the quantities y_t y k_t solve the problem of the representative firm:

$$max \quad y_t - w_t - r_t k_t$$

$$y_t = f(k_t)$$

and profits are equal to zero.

$$y_t = w_t + r_t k_t + w_t$$

- In each period, markets clear:

$$y_t = c_t + i_t$$

Including the input markets!

Social Planner's Problem

- Given $k_0 > 0$, a benevolent Social Planner solves:

$$Max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.a.$$

$$c_t + i_t = f(k_t) \quad \forall t$$

$$(1+\eta)k_{t+1} = (1-\delta)k_t + i_t \forall t$$

$$c_t, i_t \geq 0$$

The sequences c_t, i_t y k_{t+1} that result from this optimization are Pareto efficient.

Welfare Theorems

- Without distortions, such as taxes or externalities:
  1. A competitive equilibrium is Pareto efficient (first welfare theorem).
  2. For each Pareto efficient allocation, there exists a price system such that the allocation and such prices constitute a competitive equilibrium (second welfare theorem).

  Strategy: Characterize the CE and find the prices such that it is consistent with Planner's problem.

First Order Conditions

- Lagrangean of Social Planner's problem:

$$L = \sum_{t=0}^{\infty} [\beta^t u(c_t) - \lambda_{1.t}(c_t + i_t - f(k_t)) -$$
$$\lambda_{2,t}((1+\eta)k_{t+1} - (1-\delta)k_t - i_t)]$$

Why can we omit the non-negativity conditions?

First Order Conditions

- Maximizing with respect to L, we obtain the FOCs:
- Plus the transversality condition:

$$lim_{t\to\infty}\left(\frac{\lambda_{2,t}}{\lambda_{2,0}}k_{t+1}\right)=0$$

  where $\lambda_{2,t}$ represents the shadow price of a unit of capital.
- Using the first-order conditions, we can rewrite the transversality condition as:

$$\lim_{t\to\infty}\beta^t\left(\frac{u'(c_t}{u'(c_0}\right)k_{t+1}=0$$

First Order Conditions

- Doing some algebra:
- **Ecuación de Euler:**

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{f'(k_{t+1}) + (1-\delta)}{1+\eta}$$

- **Transversality condition:**

$$c_t = f(k_t) - (1+\eta)k_{t+1} + (1-\delta)k_t$$

Consumption is equal to the final output minus investment.

Characterization

- Nonlinear system of two first-order difference equations in c_t and k_t, initial condition k_0 and transversality condition.
- Prices are obtained from the firm's problem:

$$max \quad f(k_t) - w_t - r_t k_t$$

from where:

$$r_t = f'(k_t)$$

$$w_t = f(k_t) - f'(k_t)k_t$$

- The optimal paths for c_t y k_t, together with these prices, constitute a CE.

CE and SP equivalence

- To verify that we indeed have a CE, we characterize the solution of the representative household:

$$L = \sum_{t=0}^{\infty} [\beta^t u(c_t) - \lambda_{1.t}(c_t + i_t - w_t - r_t k_t) -$$

$$\lambda_{2,t}((1+\eta)k_{t+1} - (1-\delta)k_t - i_t)]$$

- From the first-order conditions and the transversality condition we have:
- **Euler's equation:**

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{r_{t+1} + (1-\delta)}{1+\eta}$$

and the **Feasibility constraint**:

$$c_t = w_t + [r_t + (1-\delta)]k_t - (1+\eta)k_{t+1}$$

CE and SP equivalence

- Using the prices that come from the firm's problem:

$$r_t = f'(k_t) \quad w_t + r_t k_t = f(k_t)$$

Therefore, the CE and the SP are equivalent.

Steady State

- A steady state is a CE in which all quantities per worker are constant over time:

$$c_{t+1} = c_t = c^*$$
$$k_{t+1} = k_t = k^*$$

  Therefore, the quantities in levels C_t y K_t grow at a rate η

- From the Euler's equation:

$$f'(k^*) = \frac{1+\eta}{\beta} - (1-\delta)$$

  Therefore, there exists a unique level for capital in steady state k^*

Extensions: endogenous labor supply

- So far, we assumed a perfectly inelastic labor supply.
- Now, we will make labor supply to be endogenous by adding a consumption-leisure decision.
- We will focus on the intensive margin of labor supply decisions.

Extensions: endogenous labor supply

- Consider the intertemporal utility:

$$U = \sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}, \frac{L_t - L_t^s}{L_t}\right)$$

L_t^s : denotes the household labor supply.

- Assume: $u_1 > 0$, $u_2 > 0$, $u_{11} < 0$, $u_{22} < 0$ y $u_{21} > 0$

Extensions: endogenous labor supply

- In intensive form, we divide all the variables by L_t:
  1. Intratemporal utility function:
  $$u(c_t, 1 - l_t) \tag{1}$$
  2. Budget constraint:
  $$c_t + i_t = w_t l_t + r_t k_t$$
  3. Production function:
  $$y_t = \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, \frac{L_t^s}{L_t}\right) = F(k_t, l_t)$$

Extensions: endogenous labor supply

- A CE is a set of sequences for the quantities c_t, l_t, i_t, y_t y k_{t+1} and prices w_t y r_t such that:

  1. i) Given $k_0 > 0$, w_t y r_t, the sequences $c_t, l_t, i_t y k_{t+1}$ solve the household's problem:

$$max \quad \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - l_t)$$

$$s.a$$

$$c_t + i_t = w_t l_t + r_t k_t$$

$$(1 + \eta) k_{t+1} = (1 - \delta) k_t + i_t$$

$$c_t, i_t \geq 0$$

$$0 \leq l_t \leq 1$$

Extensions: endogenous labor supply

1. ii) In each period t, given w_t and r_t, the quantities y_t, k_t y l_t solve the firm's representative problem.
2. iii) In each period t, markets clear:

$$y_t = c_t + i_t$$

$$max \quad y_t - w_t l_t - r_t k_t$$

$$s.t$$

$$y_t = F(k_t, l_t)$$

Extensions: endogenous labor supply

- Social Planner's Problem:

$$max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - l_t)$$

$$s.t$$

$$c_t + i_t = F(k_t, 1 - l_t)$$

$$(1 + \eta)k_{t+1} = (1 - \delta)k_t + i_t$$

With Lagrangean:

$$\text{L}{=}\textstyle\sum_{t=0}^{\infty}[\beta^t u(c_t, 1 - l_t) - \lambda_{1.t}(c_t + i_t - F(k_t, l_t) -$$

$$\lambda_{2,t}((1 + \eta)k_{t+1} - (1 - \delta)k_t - i_t)]$$

Extensions: endogenous labor supply

- First order conditions:

$$\frac{\partial L}{\partial c_t} = \beta^t u_1(c_t, 1 - l_t) - \lambda_{1,t} = 0$$

$$\frac{\partial L}{\partial l_t} = -\beta^t u_2(c_t, 1 - l_t) + \lambda_{1,t} F_l(k_t, l_t) = 0$$

$$-\lambda_{1,t} + \lambda_{2,t} = 0$$

$$\frac{\partial L}{\partial k_{t+1}} = \lambda_{t+1} F_K(k_{t+1}, l_{t+1}) - \lambda_{2,t}(1 + \eta) +_{2,t+1} (1 - \delta) = 0$$

Plus the usual transversality condition.

Extensions: endogenous labor supply

- Combining these conditions, we obtain the Euler Equation:

$$\frac{u_1(c_t, l_t}{\beta u_1(c_{t+1}, 1 - l_{t+1})} = \frac{F_K(k_{t+1}, l_{t+1}) + (1 - \delta)}{1 + \eta}$$

the feasibility constraint:

$$c_t = F(k_t, l_t) - (1 + \eta)k_{t+1} + (1 - \delta)k_t$$

and an additional static relationship:

$$u_1(c_t, 1 - l_t) = \frac{u_2(c_t, 1 - l_t)}{F_L(k_t, l_t)}$$

This implicitly defines a labor supply function that depends positively on the wage rate.

Extensions: endogenous labor supply

- The steady state is characterized by the system:

$$\frac{1}{\beta} = \frac{F_K(k^*, l^*) + (1-\delta)}{1+\eta}$$

$$c^* = F(k^*, l^*) - (n+\delta)k^*$$

$$u_1(c^*, 1-l^*) = \frac{u_2(c^*, 1-l^*)}{F_L(k^*, l^*)}$$

We can solve for c^*, k^* y l^*

Extensions: Exogenous growth and technical change

- Steady state: output per worker is constant. In the basic model, there is no long-run growth.
- We now introduce exogenous technical change that affects labor productivity.
- The production function is:

$$F(K_t, A_t L_t)$$

$$A_{t+1} = (1+g)A_t$$

A_t denotes technology level ($A_0 = 1$) and g is the exogenous rate of technical progress.

Extensions: Exogenous growth and technical change

- Divide all the quantities by $A_t L_t$ so everything is expressed in efficiency units of labor.

$$\hat{c}_t = \frac{C_t}{A_t L_t} = \frac{c_t}{A_t}$$

Production function:

$$\hat{y}_t = F(\frac{K_t}{A_t L_t}, 1) = f(\hat{k}_t)$$

Budget constraint:

$$\hat{c}_t + \hat{i}_t = \hat{w}_t + r_t \hat{k}_t, \quad con \quad \hat{w}_t = \frac{w_t}{A_t}$$

Capital law of motion:

$$(1+g)(1+\eta)\widehat{k_{t+1}} = (1-\delta)\hat{k}_t + \hat{i}_t$$

Extensions: Exogenous growth and technical change

- We also need to transform the utility function into efficiency units. For example:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

Then:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \beta^t A_t^{1-\sigma} \frac{\hat{c}_t^{1-\sigma}}{1-\sigma}$$

$$= \sum_{t=0}^{\infty} \hat{\beta}^t u(\hat{c}_t)$$

where $\hat{\beta} = \beta(1+g)^{1-\sigma}$

Extensions: Exogenous growth and technical change

- We have redefined variables such as their structure is similar to the baseline model. Therefore, the definition of CE and the FOCs are the same:

$$\frac{u'(\hat{c}_t)}{\hat{\beta}u'(\hat{c_{t+1}})} = \frac{f'(\hat{k}_{t+1}) + (1-\delta)}{(1+\eta)(1+g)}$$

Feasibility constraint:

$$\hat{c}_t = f(\hat{k}_t) - (1+\eta)(1+g)\hat{k}_{t+1} + (1-\delta)\hat{k}_t$$

- In the long run, the economy converges to a steady state, where $\hat{k}_t$ y $\hat{c}_t$ are constant.

Extensions: Exogenous growth and technical change

- In contrast with the baseline model, the quantities per worker grow at the same and at a constant rate g in the steady state (balanced growth path).
- This model features long-run growth but an exogenous rate independent of other parameters.

Extensions: Exogenous growth and technical change

- By construction, the balanced growth path of this model is consistent with Kaldor's stylized facts (1961):
  1. The growth rate of output per worker is constant and positive.
  2. The saving rate is constant (investment/output ratio).
  3. The real interest rate is constant.
  4. The share of each factor in national income is constant.

  These regularities correspond to advanced economies such as the United States.