Public Finance Lecture Notes

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1 Static Ramsey Taxation

In this first chapter, we cover the notion of what Ramsey Taxation means, first in a static environment. The intuitive idea behind Ramsey Taxation is solving a Planner's Problem conditional on a feasibility condition and also an additional one which indicates that we choose from a choice set that characterizes a competitive equilibrium (raising enough revenue from a set of available instruments). In general, solving a Ramsey problem involves the following steps:

1. Identify what is the objective function of the tax designer.

2. Identify what are the available instruments for the tax designer.

3. Mapping between the setting of instruments in step 2. and the planner utility in 1.

1.1 Environment

Consider an economy in which n consumption goods are produced using labor:

$$F(c_1 + g_1, ..., c_n + g_n, l) = 0$$

Notice that I am implicitly assuming that both goods consumed by individuals (c_i) and the government (g_i) are produced using the same technology. F is assumed to be CRS. Individuals solve the following maximization problem:

$$\underset{c_1,...,c_n,l}{Max}\, U(c_1, ..., c_n, l)$$

$$s.t.$$

$$\sum_{i=1}^{n} p_i(1 + \tau_i)c_i = l$$

[*]To prepare these notes, I have benefited from: V.V. Chari and Larry Jones 2nd year lecture notes, and from Roozbeh Hosseini notes on Public Economics. These notes are preliminary and may contain errors.

τ_i denotes the tax that is levied on good i. Regarding firms, there is a representative firm that produces goods using technology F:

$$\underset{x_1,...,x_n,l}{Max} \sum_{i=1}^{n} p_i x_i - l$$
$$s.t.$$
$$F(x_1, ..., x_n, l) = 0$$

Government finance its purchases through taxes:

$$\sum_{i=1}^{n} p_i g_i = \sum_{i=1}^{n} p_i \tau_i c_i$$

1.2 Competitive Equilibrium

Given the tax system, a competitive equilibrium in this environment is an allocation together with a price system such that:

- Given policy $\pi \equiv (\tau_1, ..., \tau_n)$ and prices $p \equiv (p_1, ..., p_n)$, (c, l) solves consumer's problem.
- Given prices p, x_i solves the problem of representative firm.
- Government budget holds.
- Markets clear:

$$c_i + g_i = x_i \quad \forall i$$

Proposition 1 *Any competitive equilibrium allocations must satisfy the feasibility condition:*

$$F(c_1 + g_1, ..., c_n + g_n) = 0 \tag{1}$$

and an implementability constraint:

$$\sum_{i=1}^{n} U_i c_i + U_l l = 0 \tag{2}$$

Furthermore, any allocation that satisfy the two equations above can be supported as a competitive equilibrium for appropriately constructed policies and prices.

Proof:

Suppose (c, x, l) is an allocation such that it constitutes part of a Competitive Equilibrium. Then, the following FOC must be satisfied:

$$U_l = -\frac{U_i}{p_i(1+\tau_i)} \quad \forall i$$

Then replacing this in the budget constraint of the individual (getting rid of the prices and taxes), we obtain the implementability constraint.

Now let's assume that we have an allocation that satisfies both the implementability and the feasibility constraint. Then, build prices as:

$$p_i = -\frac{F_i}{F_l} \quad \forall i$$

$$1+\tau_i = \frac{U_i}{U_l}\frac{F_l}{F_i} \quad \forall i$$

It is easy to check that indeed the allocation together with this price system, constitute a competitive equilibrium.

1.3 Ramsey Problem

The question we try to answer is how taxes should be. In other words, we look for a tax system that is able to maximize the welfare of individuals subject to having some tax instruments that are able to raise enough revenue with a resulting allocation that constitutes a competitive equilibrium.

This can also be seen as a game: First, the government sets a policy and then the consumers decide how much to consume and work given the government policy. We aim to find the equilibrium of such a game. Suppose that the set of feasible policies for the government is Π:

Definition 1 (Ramsey equilibrium) *A Ramsey equilibrium is a policy* $\pi = (\tau_1, ..., \tau_n) \in \Pi$*, allocation rules* $c(.)$*,* $x(.)$ *and* $l(.)$ *and price function* $p(.)$ *such that:*

$$\pi \in \underset{\pi' \in \Pi}{argmax} U(c(\pi'), l(\pi'))$$

$$s.t.$$

$$\sum_{i=1}^{n} p_i g_i = \sum_{i=1}^{n} p_i \tau_i c_i$$

and $(c(\pi'), x(\pi'), l(\pi'))$ *together with* $p(\pi')$ *is a competitive equilibrium for every* $\pi' \in \Pi$*. If the allocation and the price constitute a Ramsey equilibrium, then we call* $(c(\pi), x(\pi), l(\pi))$ *a Ramsey allocation.*

Proposition 2 *Suppose* c^* *and* l^* *are part of a Ramsey allocation. Then:*

$$(c^*, l^*) \in \underset{c,l}{argmax} U(c,l)$$

$$s.t.$$

$$\sum_{i=1}^{n} U_i c_i + U_l l = 0$$

$$F(c_1 + g_1, ..., c_n + g_n, l) = 0$$

Proof:

If (c^*, l^*) constitute a Ramsey allocation, then, there is a price $p(\pi^*)$ such that the allocation and that price is a competitive equilibrium. If that is the case, then it is going to satisfy the government budget-balance condition.If it is a competitive equilibrium, it satisfies feasibility and implementability (by Proposition 1). Now, notice also that if an allocation solves the problem in proposition two, then we can build a price such that the resulting allocation together with the price is a competitive equilibrium (and therefore satisfies the government budget-balance condition). Therefore, both problems are equivalent. In other words, both problems maximize utility with a choice set equal to the allocations that are part of a competitive equilibrium for some price system. Q.E.D.

1.4 Elasticities and Optimal Taxes

Let's assume there are only 2 goods. Ramsey problem in this case is:

$$\underset{c_1, c_2, l}{Max} U(c_1, c_2, l)$$

$$s.t.$$

$$U_1 c_1 + U_2 c_2 + U_l l = 0$$

$$F(c_1 + g_1, c_2 + g_2, l) = 0$$

Suppose λ and η are the Lagrange Multiplieres for the implementability and the feasibility condition, respectively. Then:

$$U_i + \lambda(U_i + U_{1i} c_i + U_{2i} c_2 + U_{li} l) = \eta F_i$$

$\forall i$

$$U_l + \lambda(U_l + l U_{ll} + U_{1l} c_1 + U_{2l} c_2) = \eta F_l$$

Now define:

$$\Psi_i = -\frac{U_{1i} c_1 + U_{2i} c_2 + U_{li} l}{U_i} \quad \Psi_l = -\frac{U_{1i} c_1 + U_{2i} c_2 + U_{ll} l}{U_l}$$

From these we have:

$$\frac{\eta F_i}{U_i} = 1 + \lambda - \lambda \Psi_i \quad \frac{\eta F_l}{U_l} = 1 + \lambda - \lambda \Psi_l$$

From the individual problem we had:

$$1 + \tau_i = \frac{U_i}{U_l} \frac{F_l}{F_i}$$

Matching this with the Ramsey problem:

$$1 + \tau_i = \frac{1 + \lambda - \lambda \Psi_l}{1 + \lambda - \lambda \Psi_i}$$

and this means that if $\Psi_i > \Psi_j$ then you tax more good i than good j. However, Ψ does not tell us that much per se. To understand clearly what is going on, let's go through some special cases of U.

1.4.1 Additive separable utility functions

Let's assume that U has the following functional form:

$$U(c_1, c_2, l) = u_1(c_1) + u_2(c_2) - v(l)$$

Then:

$$\Psi_i = -\frac{U_{ii} c_i}{U_i}$$

Now I will show that there is a relationship between Ψ_i and the income-elasticity of good i $\forall i$. To do this, suppose there is a non-wage income m, such that $p_1 c_1 + p_2 c_2 = l + m$. Taking FOC (ignoring the tax part) we have:

$$U_i(c_i(p, m)) = p_i \phi(p, m)$$

where $\phi(p, m)$ is the Lagrange Multiplier on the budget constraint. Deriving w.r.t. m we have:

$$U_{ii} \frac{\partial c_i}{\partial m} = p_i \frac{\partial \phi}{\partial m} = \frac{U_i}{\phi} \frac{\partial \phi}{\partial m}$$

and Rearranging:

$$\frac{U_{ii} c_i}{U_i} \frac{m}{c_i} \frac{\partial c_i}{\partial m} = \frac{m}{\phi} \frac{\partial \phi}{\partial m}$$

Define η_i as:

$$\eta_i = \frac{m}{c_i} \frac{\partial c_i}{\partial m}$$

Then:

$$\Psi_i = -\frac{m}{\phi}\frac{\partial \phi}{\partial m}\frac{1}{n_i}$$

Therefore: $H_i > H_j \iff \eta_i > \eta_j$. This means needs should be more taxed than luxury goods.

Result 1 *If preferences are additive separable, necessities should be taxed more than luxuries.*

1.4.2 Quasi-linear utility function

Suppose now preferences are as in the previous section but now it has the following peculiarity:

$$v(l) = l$$

Then, these preferences don't have income effect and therefore using income elasticities to base our analysis is not the most appropriate thing to do anymore. Instead, we use now price elasticities. Again, taking FOC we have:

$$U_i(c_i) = p_i\phi$$

and deriving w.r.t. p_i:

$$U_{ii}\frac{\partial c_i}{\partial p_i} = \phi = \frac{U_i(c_i)}{p_i}$$

Then:

$$U_{ii}\frac{\partial c_i}{\partial p_i}\frac{p_i}{c_i} = \frac{U_i(c_i)}{c_i}$$

Therefore:

$$\Psi_i = \frac{1}{\epsilon_i}$$

Therefore, you tax more the less elastic goods.

Result 2 *If preferences are additive-separable and quasi-linear, price-inelastic goods should be taxed more.*

1.5 Uniform Commodity Taxation

This is one of the most useful and powerful results from Ramsey taxation. Let's see why. First, suppose that preferences are weakly separable in consumption and leisure:

$$U(c_1, ..., c_n, l) = W(G(c_1, ..., c_n), l) \tag{3}$$

let's assume also that G(.) is homothetic.

Proposition 3 *Suppose that preferences satisfy (3). Then, it is optimal to tax all the goods at the same rate:* $\tau_i = \tau_j \quad \forall i, j.$

Proof:

Homotheticity of $G(.)$ implies:

$$\frac{U_i(\alpha c, l)}{U_j(\alpha c, l)} = \frac{U_i(c, l)}{U_j(\alpha, l)}$$

which means:

$$U_i(\alpha c, l) = U_j(\alpha c, l)\frac{U_i(c, l)}{U_j(c, l)}$$

Taking derivative with respect to α and setting $\alpha = 1$ we obtain:

$$\frac{\sum_{k=1}^{n} U_{ik} c_k}{U_i} = \frac{\sum_{k=1}^{n} U_{jk} c_k}{U_j}$$

also notice that:

$$U_l = W_l, \quad U_{li} = W_{lg} G_i \quad U_i = W_g G_i$$

Therefore:

$$\Psi_i = \frac{-\sum_{k=1}^{n} U_{ik} c_k}{U_i} - \frac{U_{il} l}{U_i} = \frac{-\sum_{k=1}^{n} U_{ik} c_k}{U_i} - \frac{W_{lg} G_i l}{W_g G_i}$$
$$= \frac{-\sum_{k=1}^{n} U_{ik} c_k}{U_i} - \frac{W_{lg} l}{W_g} = H_j$$

Q.E.D.

1.6 Intermediate goods taxation

Another important result from static Ramsey taxation is that intermediate goods shall not be taxed. To formalize the idea, suppose there are 2 sectors: One sector produces commodity x_1 that is consumed both by the households c_1 and by the government g_1. To produce commodity 1, firms need to combine an intermediate good z and labor l_1 according to the following production function:

$$f(x_1, z, l_1) = 0,$$

The other sector, uses labor l_2 as input to produce x_2 that can be used as an input in production of good x_1 (z) or it can be consumed (c_2, g_2). This is done with the following technology:

$$h(x_2, l_2) = 0,$$

What is the problem that households solve?

$$\underset{c,l}{Max} \quad U(c_1, c_2, l)$$

$$s.t.$$

$$p_1(1+\tau_1)c_1 + p_2(1+\tau_2)c_2 \leq l_1 + l_2$$

Producer of good 1 solves:

$$\underset{x_1,z,l_1}{Max} p_1 x_1 - l_1 - p_2(1+\tau_z)z$$

$$s.t.$$

$$f(x_1, z, l_1) = 0,$$

The FOC from this problem is:

$$\lambda_1 f_l = 1$$

$$p_2(1+\tau_z) = \lambda_1 f_z$$

Then:

$$\frac{f_z}{f_l} = p_2(1+\tau_z)$$

Then:

$$\frac{f_z}{f_l} = p_2(1+\tau_z)$$

Producer of the good 2 solves:

$$\underset{x,l_2}{Max} \quad p_2 x_2 - l_2$$

$$s.t.$$

$$h(x_2, l_2) = 0$$

The following condition characterizes this firm's problem:

$$p_2 = \lambda_2 h_{x_2}$$

$$-1 = \lambda_2 h_l$$

Therefore:

$$\frac{h_x}{h_l} = -p_2$$

Combining the conditions of both sectors:

$$\frac{f_z}{f_l} = -\frac{h_x}{h_l}(1+\tau_z)$$

The government budget constraint is:

$$\tau_z z + \tau_1 c_1 + \tau_2 c_2 = p_1 g_1 + p_2 g_2$$

Feasibility implies:

$$c_1 + g_1 = x_1$$

$$c_2 + g_2 + z = x_2$$

$$f(x_1, l_1, z) = 0$$

$$h(x_2, l_2) = 0$$

What is the implementability constraint in this environment?

From the first order conditions of the consumer we have:

$$\frac{U_{c_1}}{p_1(1+\tau_1)} = \frac{U_{c_2}}{p_2(1+\tau_2)} = U_{l_1+l_2}$$

Replacing the prices and taxes to obtain expressions only as a function of the allocation, we obtain the following implementability constraint:

$$U_{c_1} c_1 + U_{c_2} c_2 + U_l(l_1 + l_2) = 0$$

Therefore, the Ramsey problem is:

$$\underset{c_1, c_2, l_1+l_2}{Max} \; U(c_1, c_2, l_1 + l_2)$$

$$s.t.$$

$$U_{c_1} c_1 + U_{c_2} c_2 + U_l(l_1 + l_2) = 0 \qquad (\lambda)$$

$$f(c_1 + g_1, z, l_1) = 0, \qquad (\phi_1)$$

$$h(c_2 + g_2 + z, l_2) = 0, \qquad (\phi_2)$$

Taking first order condition w.r.t. z:

$$\phi_1 f_z = -\phi_2 h_z$$

Then:

and if we take derivative with respect to labor:

$$f_l \phi_1 = U_l + \lambda \left(U_{ll}(l_1 + l_2) + U_l + U_{cl} c\right)$$

$$h_l \phi_2 = U_l + \lambda \left(U_{ll}(l_1 + l_2) + U_l + U_{cl} c\right)$$

Then:

$$f_l \phi_1 = h_l \phi_2$$

Therefore:

$$\frac{f_z}{h_z} = -\frac{f_1}{h_1}$$

Then $\tau_z = 0$ and not distort production efficiency. The MRT's are not distorted in the planner's problem!

Exercise: Give an example where this result does not hold. Hint: Suppose you can't tax all the consumption goods.

1.7 A last result: Lump sum taxes are awesome but never used...

To illustrate this result, consider an environment with two households: $i = 1, 2$. Suppose there are J goods and $c_{i,j}$ denotes consumption of good j of individual i.

Assume endowments are given by labor endowments of n^i for $i = 1, 2$. There is a production function $F(n)$ such that a feasible allocation satisfies:

$$\sum_i \sum_j c_{i,j} + g \leq F(n)$$

1.7.1 Unconstrained Planner's Problem

A Social Planner would solve:

$$\underset{c_{j,i}, n_i}{Max} \quad \mu u^1(c_{1,1}, ..., c_{1,J}; 1 - n_1) + (1 - \mu) u^2(c_{2,1}, ..., c_{2,J}; 1 - n_2)$$

$$s.t.$$

$$\sum_i \sum_j c_{i,j} + g \leq F(n_1 + n_2)$$

From the First-Order conditions we have:

$$c_{1,j} : \quad \mu u^1_j = \lambda$$

$$n_1 : \quad \mu u_l^1 = \lambda F'$$

$$c_{2,j} : \quad \mu u_j^2 = \lambda$$

$$n_2 : \quad \mu u_l^2 = \lambda F'$$

The following equations characterize the solution to this problem:

$$\frac{u_l^1}{u_j^1} = F' \quad \forall j$$

$$\frac{u_l^2}{u_j^2} = F' \quad \forall j$$

$$\sum_i \sum_j c_{i,j} + g = F(n_1 + n_2)$$

$$\mu u_j^1 = (1 - \mu) u_j^2$$

1.7.2 What is a Tax-Distorted Competitive Equilibrium (TDCE) with Lump Sum Taxes?

A TDCE is:

- Prices and wages: $p_1, ..., p_J, p_g, w$.
- Taxes $T^i(p_1 c_1^i, ..., p_J c_J^i, wn^i)$ (a very general form for taxes). Lump sum taxes would be $T^i(p_1 c_1^i, ..., p_J c_J^i, wn^i) = T^i$.
- Allocations for households and firms that solve:

$$\underset{c_{i,j};1-n^i}{Max} \quad u^i(c_{i,1}, ..., c_{i,J}; 1 - n^i)$$

$$\sum_j p_j c_{i,j} \leq wn^i - T^i(p_1 c_{i,1}, ..., p_J c_{i,J}, wn^i) + \pi^i \quad (\eta)$$

and

$$\underset{c_j^f, g^f, n^f}{Max} \quad \sum_j p_j c_j^f + p_g g^f - wn^f$$

$$s.t.$$

$$\sum_j c_j^f + c_g^f \leq F(n^f)$$

- Market clearing:

$$c_{1,j} + c_{2,j} = c_j^f$$

$$g = c_g^f$$

$$n_1 + n_2 = n^f$$

$$T^1 + T^2 = p_g g$$

Assume F has CRS (what does it imply for $\pi's$? The firm's FOCs imply $p_j = p_{j'} = p_g$ for all interior equilibria. Let's call this common price p. Also:

$$pF' = w$$

The Household FOCs are:

$$c_{i,j}: \quad u^i_j = \eta^i[p + pT_{i,j}] = \eta^i p[1 + T_{i,j}],$$

$$n_i: \quad u^i_l = \eta^i[w - wT^i_l] = \eta^i w[1 - T^i_l]$$

then:

$$\frac{u^i_l}{u^i_j} = \frac{\eta^i w[1 - T^i_l]}{\eta^i p[1 - T^i_j]} = \frac{F'[1 - T^i_l]}{1 + T^i_j}$$

1.7.3 Implementation of the Planner's Problem

From the TDCE we have:

$$\frac{u^i_l}{u^i_j} = \frac{\eta^i w[1 - T^i_l]}{\eta^i p[1 - T^i_j]} = \frac{F'[1 - T^i_l]}{1 + T^i_j}$$

and from the Planner's problem we have:

$$\frac{u^i_l}{u^i_j} = F'$$

Combining these conditions we have that in order to implement Planner's problem:

$$\frac{1 - T^i_l}{1 + T^i_j} = 1$$

The easiest way to implement this condition is by making $T^i(p_1 c_{i,1}, ..., p_J c_{i,J}, wn_i) = T^i$ for example.

Exercise: Show that this result still holds if g enters the utility function. Does it still hold if g enters the production function?

2 Dynamic Ramsey

For the purposes of this course, we will focus on a deterministic environment. Our main goal will be deriving the Chamley-Judd result (Chamley (1986), Judd (1985). If you are interested in studying a stochastic business-cycles environment please read Chari and Kehoe (1998).

Our environment is the following: We have a continuous of identical individuals (representative agent) and the government has to raise revenue to finance expenditure g_t. Government does it by levying distortionary taxes/subsidies on consumption, on investment, on labor and on capital income[1]. We assume government can also issue debt. Each individual thus solves:

$$\underset{c_t,l_t,x_t,k_{t+1},b_{t+1}}{Max} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

$$(1+\tau_{c,t})c_t + (1+\tau_{x,t})x_t + b_{t+1} \leq (1-\tau_{l,t})w_t l_t + (1-\tau_{k,t})r_t k_t + R_{b,t} b_t \quad (\lambda_t)$$

$$k_{t+1} \leq (1-\delta)k_t + x_t$$

$$-b_{t+1} \leq B$$

$$k_0, b_0 \quad given$$

B is an arbitrary large number. By taking the FOC's we have:

$$c_t: \quad \beta^t U_{ct} - \lambda_t(1+\tau_{c,t}) = 0,$$

$$l_t: \quad \beta^t U_{lt} + \lambda_t(1+\tau_{l,t})w_t = 0,$$

$$k_{t+1}: \quad \lambda_{t+1}(1-\tau_{k,t+1})r_{t+1} - \lambda_t(1+\tau_{x,t}) + \lambda_{t+1}(1+\tau_{x,t+1})(1-\delta) = 0,$$

$$b_{t+1}: \quad \lambda_{t+1}R_{b,t+1} - \lambda_t = 0,$$

The government budget balance is:

$$g_t + R_{bt}b_t = b_{t+1} + \tau_{xt}x_t + \tau_{ct}c_t + \tau_{lt}w_t l_t + \tau_{kt}r_t k_t,$$

and the feasibility constraint:

$$c_t + x_t + g_t = F(k_t, l_t)$$

Competitive prices imply:

$$r_t = F_{k,t}, \quad w_t = F_{l,t}$$

[1] Notice below that we are implicitly assuming that the government only has access to linear taxes as its set of instruments

Then, a TDCE is an allocation $X = \{c_t, l_t, b_{t+1}, k_{t+1}, x_t\}_{t=0}^{\infty}$, a price system $\{r_t, w_t, R_{bt}\}_{t=0}^{\infty}$ and a government policy $\pi = \{\tau_{ct}, \tau_{lt}, \tau_{xt}, \tau_{kt+1}\}_{t=0}^{\infty}$, such that given the prices and government policy, the allocations solve the consumer's problem, prices are competitive, government budget holds and allocations are feasible (market clearing).

What is a Ramsey Equilibrium in this context?

A Ramsey equilibrium is a policy π, an allocation rule $x(.)$, and price rules $r(.), w(.)$ and $R_b(.)$ such that:

$$\pi \in \underset{\pi'}{argmax} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

$$s.t.$$

$$g_t + R_{bt} b_t = b_{t+1} + \tau_{xt} x_t + \tau_{ct} c_t + \tau_{lt} w_t l_t + \tau_{kt} r_t k_t,$$

$x(\pi)$ be a competitive equilibrium, and for any policy π', allocation $x(\pi')$ and prices $(r(\pi'), w(\pi'), R_b(\pi')$ be a competitive equilibrium.

Now we derive our implementability condition. We will assume the transversality condition holds. In most of the cases, this will actually be the case since the conditions of Ekeland and Sheikman (1986) and/or Weitzman (1973) are satisfied:

$$\lim_{t \to \infty} \lambda_t b_{t+1} = 0,$$

$$\lim_{t \to \infty} \lambda_t k_{t+1} = 0,$$

Multiply budget constraint by (λ_t), sum over t and use the transversality conditions:

$$\sum_{t=0}^{\infty} \lambda_t [(1 + \tau_{ct}) c_t + (1 + \tau_{xt})(k_{t+1} - (1 - \delta) k_t) + b_{t+1}] = \sum_{t=0}^{\infty} \lambda_t [(1 - \tau_{lt}) w_t l_t + (1 - \tau_{kt}) r_t k_t + R_{bt} b_t]$$

Use government budget constraint and first-order conditions to obtain:

$$\sum_{t=0}^{\infty} \lambda_t [(1 + \tau_{ct}) c_t - (1 - \tau_{lt}) w_t l_t] = \lambda_0 \{[(1 + \tau_{x0})(1 - \delta) + (1 - \tau_{k0}) r_0] k_0 + R_{b0} b_0\}$$

This can be additionally expressed as:

$$\sum_{t=0}^{\infty} \beta^t [U_{ct} c_t + U_{lt} l_t] = U_{c0} \{[(1 + \tau_{x0})(1 - \delta) + (1 - \tau_{k0}) r_0] k_0 + R_{b0} b_0\} \tag{4}$$

The idea is the following: we want to obtain a condition that can tell us that an allocation is indeed a TDCE. We want this condition as a function of only the allocation (no prices no taxes). In other words, the Ramsey problem tries to find what is the optimal policy in terms of maximizing utility subject to raising enough resources for the government and with such policy inducing a TDCE. This last condition is captured in the implementability condition.

Proposition 4 *A feasible allocation* $x = \{c_t, l_t, b_{t+1}, k_{t+1}, x_t\}_{t=0}^{\infty}$ *is a TDCE if and only if it satisfies the implementability condition (4) (for some period zero policies).*

Proof:

If direction:

Suppose a feasible allocation x^* satisfies (4) for some policy. In any competitive equilibrium, bonds holding must satisfy:

$$b_{t+1} = \sum_{s=t+1}^{\infty} \beta^{t-s} \frac{[U_{cs}c_s + U_{ls}l_s]}{U_{ct}} - k_{t+1}$$

To obtain this condition, use a similar procedure that was used to obtain the implementability constraint and use the non-arbitrage condition. Given this equation, it can be seen that c_t^*, k_{t+1}^* and l_t^* uniquely identifies a sequence for b_t that is a part of the competitive equilibrium. Rate of return of capital and wages would be simply marginal products of capital and labor, respectively. Finally, we choose our taxes sequences such that:

$$\frac{1-\tau_{lt}}{1+\tau_{ct}} = -\frac{U_{lt}^*}{F_{lt}^* U_{ct}^*}$$

$$(1+\tau_{xt})\frac{U_{ct}^*}{1+\tau_{ct}} = \beta \frac{U_{ct+1}^*}{1+\tau_{ct+1}}[(1-\tau_{xt+1})(1-\delta) + (1-\tau_{kt+1})F_{kt+1}^*]$$

$$\frac{U_{ct}^*}{1+\tau_{ct}} = \beta \frac{U_{ct+1}^*}{1+\tau_{ct+1}} R_{bt+1}$$

The only if part was proved before already.

2.1 Ramsey Problem

The Ramsey problem is the following:

$$\underset{c_t, k_{t+1}, l_t}{Max} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} \beta^t [U_{ct}c_t + U_{lt}l_t] = U_{c0}\{[(1+\tau_{x0})(1-\delta) + (1-\tau_{k0})r_0]k_0 + R_{b0}b_0\} \qquad (\lambda)$$

$$c_t + g_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t \qquad (\mu_t)$$

To make our lives easier define:

$$W(c, l, \lambda) = U(c, l) + \lambda[U_c c + U_l l]$$

so we can write our Ramsey problem as:

$$\underset{c_t,k_{t+1},l_t}{Max} \sum_{t=0}^{\infty} W(c_t, l_t, \lambda)$$

$$s.t.$$

$$c_t + g_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t \qquad (\phi_t)$$

Now, by taking First Order Conditions we have:

$$\frac{W_{lt}}{W_{ct}} = -F_{lt}$$

$$\frac{W_{ct}}{W_{ct+1}} = \beta(1 - \delta + F_{kt}) \quad \text{for } t \geq 1 \tag{5}$$

2.2 Chamley-Judd Result

Proposition 5 *If the solution to the Ramsey Problem converges to a steady state, then at the steady state, the tax rate on capital income is zero.*

Proof:

In (5), we have that in steady state:

$$1 = \beta(1 - \delta + F_k)$$

which implies that there is no intertemporal distortion in the long-run. Comparing this with the Euler-Equation of the TDCE:

$$\frac{(1+\tau_{xt})(1+\tau_{ct+1})}{(1+\tau_{ct})(1+\tau_{xt+1})} = \beta\left[1 - \delta + \left(\frac{1-\tau_{kt+1}}{1+\tau_{xt+1}}\right)F_{kt+1}\right]$$

So any allocation that satisfies the implementability constraint can be implemented by using only 2 out of the 4 taxes. In particular, we only need a constant consumption tax and $\tau_{kt+1} = 0$.

Q.E.D.

Lucas said 'One principle of Ramsey taxation is that taxes should be spread evenly over similar goods... Since capital taxation... involves taxing future consumption at higher rates than early consumption...capital is a bad thing to tax'.

2.3 Heterogeneous consumers

Suppose there are two types of consumers $i \in \{1, 2\}$ with preferences:

$$\sum_{t=0}^{\infty} \beta^t U^i(c_{it}, l_{it})$$

Then the resource constraint for this economy is:

$$c_{1t} + c_{2t} + k_{t+1} = F(k_t, l_{1t}, l_{2t}) + (1 - \delta)k_t$$

and the implementability constraint for each individual would be:

$$\sum_{t=0}^{\infty} \beta^t [U^i_{ct} c_{it} + U^i_{lt} l_{it}] = U^i_0 \left\{ [(1 + \tau_{x0})(1 - \delta) + (1 - \tau_{k0}) r_0] k^i_0 + R_{b0} b^i_0 \right\}$$

The Planner's problem does not need to be utilitarian. Let's say the Planner puts welfare weights ω_i on consumers of type i. Then the Ramsey Problem is:

$$Max \quad \omega_1 \sum_{t=0}^{\infty} \beta^t U^1(c_{1t}, l_{1t}) + \omega_2 \sum_{t=0}^{\infty} \beta^t U^2(c_{2t}, l_{2t})$$

subject to the resource constraint and the implementability constraint.

Let λ_i denote the Lagrange Multiplier associated with the implementability constraint of individuals of type i and write:

$$Max \sum_{t=0}^{\infty} \beta^t W(c_{1t}, c_{2t}, l_{1t}, l_{2t}, \lambda_1, \lambda_2)$$

$$s.t.$$

$$c_{1t} + c_{2t} + k_{t+1} = F(k_t, l_{1t}, l_{2t}) + (1 - \delta)k_t \quad (\phi_t)$$

with

$$W(c_1, c_2, l_1, l_2, \lambda_1, \lambda_2) = \sum_{i=1,2} [\omega_i U^i(c_i, l_i) + \lambda_i (U^i_c c_i + U^i_l l_i]$$

By taking first order conditions we obtain:

$$W_{cit} = \beta W_{cit+1} (1 - \delta + F_{kt+1},$$

And therefore in the steady state:

$$1 = \beta(1 - \delta + F_{kt+1}),$$

Thus: tax on capital should be zero in the steady state.

2.4 Capitalists vs Workers (Judd 1985)

Define a 'worker' as follows: It is a type of consumer (let's say type 1) and this type of consumer does not hold any asset and cannot save, invest or borrow. Define 'capitalists' as the type of individuals (type 2) that holds of the capital of the economy. In this case, the implementability constraint for a 'worker' will be:

$$U^1_{ct}c_{1t} + U^1_{lt}l_{1t} = 0 \quad \forall t$$

$$\sum_{t=0}^{\infty} \beta^t [U^2_{ct}c_{2t}] = U^2_0 \left\{ [(1+\tau_{x0})(1-\delta) + (1-\tau_{k0})r_0]k^2_0 + R_{b0}b^2_0 \right\}$$

Suppose the Planner only cares about workers (that means, the Pareto weight for capitalists is zero):

$$Max \sum_{t=0}^{\infty} \beta^t U^1(c_{1t}, l_{1t})$$

$$s.t.$$

$$U^1_{ct}c_{1t} + U^1_{lt}l_{1t} = 0 \quad \forall t$$

$$\sum_{t=0}^{\infty} \beta^t U^2_{ct}c_{2t} = U^2_0 \left\{ [(1+\tau_{x0})(1-\delta) + (1-\tau_{k0})r_0]k^2_0 + R_{b0}b^2_0 \right\}$$

$$c_{1t} + c_{2t} + k_{t+1} = F(k_t, l_{1t}, l_{2t}) + (1-\delta)k_t, \quad (\phi_t)$$

and define:

$$W(c_1, c_2, l_1, l_2, \lambda_1, \lambda_2) = U^1(c_1, l_1) + \lambda_i(U^1_c c_1 + U^1_l l_1)$$

First Order Conditions imply:

$$\lambda \beta^t [U^2_{cct}c_{2t} + U^2 c_t] + \phi_t = 0,$$

$$\phi_t = \phi_{t+1}(1 - \delta + F_{kt+1})$$

in steady state $\phi_{t+1} = \beta\phi_t$ and then:

$$1 = \beta(1 - \delta + F_{kt+1})$$

So Chamley-Judd's result holds.

2.5 Dividend Taxes (Mcgrattan and Prescott - 2005)

In this example, we will incorporate corporate taxes and dividend taxes. In this environment, individuals can trade share of corporations, s_t, at price v_t. Denote dividends by d_t. Dividends are taxed at the rate τ_{dt}. Consumers solve:

$$\sum_{\substack{t=0 \\ c_t, s_{t+1}, l_t}}^{\infty} \beta^t U(c_t, l_t)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} p_t[c_t + v_t(s_{t+1} - s_t)] \leq \sum_{t=0}^{\infty} p_t[(1-\tau_{dt})d_t s_t + (1-\tau_{lt})w_t l_t]$$

$$s_0 = 1$$

By taking First Order Conditions we have:

$$\frac{U_{ct}}{U_{lt}} = -(1-\tau_{lt})w_t$$

$$p_t v_t = p_{t+1} v_{t+1} + p_{t+1}(1-\tau_{dt+1})d_{t+1}$$

Then, the implementability constraint will be:

$$\sum_{t=0}^{\infty} \beta^t [U_{ct} c_t + U_{lt} l_t] = U_{c0}[v_0 + (1-\tau_{d0})d_0]s_0 \tag{6}$$

In the firm's side, there is a corporation that maximizes the present discounted value of owner's dividends and pays taxes τ_t on corporate income;

$$Max \sum_{t=0}^{\infty} p_t(1-\tau_{dt})d_t$$

$$s.t.$$

$$d_t = f(k_t, l_t) - x_t - w_t l_t - \tau_t(f(k_t, l_t) - \delta k_t - w_t l_t),$$

$$k_{t+1} = (1-\delta)k_t + x_t,$$

First order condition for the corporation is:

$$f_{lt} = w_t$$

$$\frac{p_t(1-\tau_{dt})}{p_{t+1}(1-\tau_{dt+1})} = 1 + (1-\tau_{lt+1})(f_{kt+1} - \delta)$$

The feasibility conditions are:

$$c_t + k_{t+1} + g_t = f(k_t, l_t) + (1-\delta)k_t,$$

$$s_t = 1$$

and the government budget constraint:

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t[\tau_{dt} d_t s_t + \tau_t(f(k_t, l_t) - \delta k_t - w_t l_t) + \tau_{lt} w_t l_t]$$

Stop for a second and think: Is the implementability constraint (6) sufficient? This means, can any competitive equilibrium be supported by (6)? If not, what other conditions should be added?

2.6 Exercise for you

Now suppose we are interested in studying a non-steady state situation and we have that preferences are given by:

$$U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - v(l),$$

What is the Ramsey tax on capital income for $t \geq 2$?

2.7 Werning (QJE-2007)

In this paper, Werning studies a dynamic environment in which indiivduals are heterogeneous in terms of their skills. In this case, the government does not rule out lump-sum taxation. Instead, it allows it. However, the government is not allowed to tax the individual's skills. Government is only allowed to tax labor income (distortionary tax). This tax can be used to redistribute income across people with different skill.

2.7.1 Environment

- Denote by c and l consumption and hours worked, respectively. Labor productivity is determined by θ, meaning that working l hours delivers θl efficiency labor units. Preferences are represented by a utility function $U^i(c, l)$ that can be written as: $U^i\left(c, \frac{y}{\theta^i}\right)$.

- There is a finite number of types $\theta \in \Theta = \{\theta^1, ..., \theta^N\}$. If an individual is of type θ^i we will just call them the i-type individual. The fraction of individuals of type i in the population is given by π^i.

- The aggregate state of the economy is $s_t \in S$ with S a finite set and publicly observable. The history of states at period t is denoted by $s^t = (s_0, s_1, ...s_t$ and the probability of a history s^t is $Pr(s^t)$.

- Production is done using capital and labor.

2.7.2 Individual's Problem

An i-type individual solves the following maximization:

$$\underset{c,y}{Max} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t Pr(s^t) U^i(c_t(s^t), y(s^t))$$

s.t.

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t)[c_t(s^t) + k_{t+1}(s^t)] \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t)[w_{s^t}(1-\tau(s^t))y(s^t) + R(s^t)k(s^{t-1})] - T$$

$$k_0^i(s_0) = k_0^i \quad \text{given}$$

where $R_t(s^t) = 1 + (1-\kappa(s^t))(r_t(s^t) - \delta)$ and $T = \sum_{s^t \in S^t} p(s^t)T(s^t)$ denotes the present value of lump-sum taxes. There is heterogeneity in skills and initial endowment of capital.

2.7.3 Markets and Government

Denote the aggregate variables by:

$$L_t(s^t) = \sum_i \pi^i y_t^i(s^t), \quad C(s^t) = \sum_i \pi^i c_t^i(s^t), \quad K_t(s^t) = \sum_i \pi^i k_t^i(s^t)$$

Then an allocation is said to be feasible if it satisfies:

$$C_t(s^t) + K_t(s^t) + g_t(s^t) = F(K_t(s^{t-1}), L_t(s^t) + (1-\delta)K_t(s^{t-1})$$

2.7.4 Firms

Firms optimization conditions will just imply:

$$r_t(s^t) = F_k(K_t(s^{t-1}), L_t(s^t)), \quad L_t(s^t) = F_l(K_t(s^{t-1}), L_t(s^t)),$$

Define a competitive equilibrium for this environment

2.7.5 Implementability Constraint

Werning (2007) exploits the fact that labor income taxes are uniform across types. Furthermore, he shows that implementability constraint can be written just as a function of the aggregates. To see this, notice that from the First-Order-Conditions we have:

$$\frac{U_y^i(s^t)}{U_c^i(s^t)} = \frac{U_y^j(s^t)}{U_c^j(s^t)} = -w(s^t)(1-\tau(s^t)) \quad \forall i, j,$$

$$\frac{U_c^i(s^t)}{U_c^i(s^0)} = \frac{U_c^j(s^t)}{U_c^j(s^0)} = \frac{p(s^t)}{\beta^t Pr(s^t)p(s_0)} \quad \forall i,j,$$

An important feature to notice here is that with linear taxation, all workers face the same after-tax prices for consumption $p(s^t)$, and labor $-p(s^t)w(s^t)(1-\tau(s^t)$. As a result, marginal rates of substitution are equated across workers. This means that any equilibrium delivers an efficient assignment of individual consumption and labor. More specifically, all inefficiencies that arise due to distortive taxation are confined to the determination of aggregates $\{c(s^t), L(s^t)\}$. To see this, we will show that given aggregate consumption and labor output $(C_t(s^t), L_t(s^t)$, the assignment of allocation of consumption $\{c^i(s^t), y^i(s^t)\}$ solves a Planning problem for some pareto weights denoted by $\varphi = \{\varphi^1, ..., \varphi^N\}$, such that $\sum_i \varphi^i \pi^i = 1$. In particular $\{c_t^i(s^t), y^i(s^t)\}$ is the solution to:

$$U^m(C(s^t), L(s^t); \varphi) \equiv \underset{c^i, y^i}{Max} \sum_i \pi^i \varphi^i U^i(c^i, y^i)$$

$$s.t.$$

$$\sum_i = \pi^i c^i = C(s^t) \quad \sum_i \pi^i y^i = L(s^t);$$

Now denote the solution of this problem by:

$$c^i = h_c^i(C, L, \varphi), \quad y^i = h_y^i(C, L, \varphi);$$

therefore

$$(c^i(s^t), y^i(s^t)) = h^i(C, L, \varphi)$$

in which $h^i = (h_c^i, h_y^i)$. Envelope condition implies:

$$U_C^m(C(s^t), L(s^t); \varphi) = \varphi^i U_c^i(c^i, y^i);$$

$$U_L^m(C(s^t), L(s^t); \varphi) = \varphi^i U_y^i(c^i, y^i)$$

Equilibrium after-tax prices can be computed as if the economy were populated by a fictious representative-agent with the utility function $U^m(c, L; \varphi)$. In any competitive equilibrium we will have then:

$$\frac{U_L^m(s^t)}{U_C^m(s^t)} = -w(s^t)(1-\tau(s^t)), \tag{7}$$

$$\frac{U_c^m(s^t)}{U_c^m(s_0)} = \frac{p(s^t)}{\beta^t Pr(s^t)p(s_0)} \quad \forall i,j \tag{8}$$

and by the envelope condition, equations (7) and (8) will hold with U^i in place of U^m and therefore marginal rates of substitution will be equated to after-tax prices. How about the implementability constraint? If we compute the individual-implementability constraint we have:

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\left[U_c^i(c_t^i(s^t),y_t^i(s^t))c_t^i(s^t)+U_y^i(c_t^i(s^t),y_t^i(s^t)\right]=U_c^i(c_0^i(s_0),y_0^i(s_0))[R_0k_0^i-T]$$

We can replace these individual allocations with functions of aggregate variables:

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t[U_C^m(C(s^t),L(s^t);)h_c^i(C(s^t),L(s^t);\varphi)+U_L^m(C(s^t),L(s^t);\varphi)h_y^i(C(s^t),L(s^t);\varphi)]$$
$$=U_c^i(C(s_0),L(s_0);\varphi)[R_0k_0^i-T]\quad\forall i \tag{9}$$

Notice that this last equation is fully expressed as a function of aggregates, weights and initial endowments.

Proposition 6 *Given initial wealth $R_0k_0^i$, an aggregate allocation $\{C_t(s^t),L_t(s^t),K_t(s^t)\}$ can be implemented as a competitive equilibrium if and only if.*

- *It is feasible.*
- *$\exists$ weights φ and lump-sum T such that the implementability constraint (9) holds $\forall i=1,...,N$.*

Proof:

We just have showed that any competitive equilibrium is feasible and it satisfies (9). Now, suppose we have a feasible allocation that satisfies (9). Then, given that allocation we can generate prices and individual allocations using (7) and (8). Since these allocations satisfy individual's optimal conditions then we have a competitive equilibrium. Furthermore, these individual allocations will be feasible since they satisfy (9).

Q.E.D.

2.7.6 A Planning Problem

Suppose the Planner weights each individual i with λ_i. $\sum_i\pi_i\lambda_i=1$:

$$max\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{i\in I}\lambda^i\pi^i\beta^tPr(s^t)U^i(h^i(C(s^t),L(s^t));\varphi)$$
$$s.t.$$
$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t[U_C^m(C(s^t),L(s^t);)h_c^i(C(s^t),L(s^t);\varphi)+U_L^m(C(s^t),L(s^t);\varphi)h_y^i(C(s^t),L(s^t);\varphi)]$$
$$=U_c^i(C(s_0),L(s_0);\varphi)[R_0k_0^i-T]\quad ...(\mu_i)\quad\forall i$$
$$C(s^t)+K(s^t)+g(s^t)=F(K_t(s^{t-1}),L_t(s^t))+(1-\delta)K_t(s^{t-1})$$

As usual we create the following variable:

$$W(C,L;\varphi,\mu,\lambda)\equiv\sum_i\pi^i\left(\lambda^iU^i(h^i(C,L;\varphi))\right)$$

$$+\mu^i\left[U_C^m(C,L;\varphi)h_c^i(C,L;\varphi)+U_L^m(C,L;\varphi)h_y^i(C,L;\varphi)\right]$$

so the problem can be written as:

$$Max\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{i\in I}\lambda^i\pi^i\beta^t Pr(s^t)W(C(s^t),L(s^t);\varphi,\mu,\lambda)-U_c^i(C(s_0),L(s_0);\varphi)\sum_i\pi^i\mu^i[R_0k_0^i-T]$$

$$s.t.$$

$$C(s^t)+K(s^t)-(1-\delta)K(s^{t-1})=F(K(s^{t-1}),L(s^t),s^t,t)$$

The first order conditions are:

$$F_L(K(s^{t-1}),L(s^t),s^t,t)=-\frac{W_C(C(s^t),L(s^t);\varphi,\mu,\lambda)}{W_L(C(s^t),L(s^t);\varphi,\mu,\lambda)}$$

$$W_C(C(s^t),L(s^t);\varphi,\mu,\lambda)=\beta\sum_{s^{t+1}\setminus s^t}W_C\left(C(s^{t+1}),L(s^{t+1});\varphi,\mu,\lambda\right)R^*(s^{t+1})Pr(s^{t+1})$$

with $R^*(s^{t+1})=1+\delta+F_k(K(s^t),L(s^{t+1}),s^{t+1},t+1)$. And the FOC with respect to initial capital:

$$\sum_i\mu^i\pi^ik_0^i=0\quad\text{or}\quad R_0=0$$

2.7.7 Optimal Taxes

By matching the Euler Equation in the Ramsey's problem with the FOC in the positive economy we have that:

$$\tau_t^*(s^t)=1-\frac{U_L^m(C;L;\varphi)}{W_L(C,L;\varphi,\mu\lambda)}\frac{W_C(C,L;\varphi,\mu\lambda)}{U_C^m(C;L;\varphi)}$$

Intertemporal optimality in equilibrium implies:

$$U_C^m(C(s^t),L(s^t);\varphi)=\beta\sum_{s^{t+1}\setminus s^t}U_C^m(C(s^{t+1}),L(s^{t+1});\varphi)R(s^{t+1})Pr(s^{t+1})$$

One way to obtain what is desired by the Ramsey planner is by making the capital income tax such that:

$$R(s^{t+1})=R^*(s^{t+1})\frac{U_C^m(C(s^t),L(s^t);\varphi)}{W_C(C(s^t),L(s^t);\varphi,\mu\lambda)}\frac{W_C(C(s^{t+1}),L(s^{t+1});\varphi,\mu\lambda)}{U_C^m(C(s^{t+1}),L(s^{t+1});\varphi)}$$

2.7.8 An Example

Suppose preferences were represented by the following felicity function:

$$U^i(c,y)=\frac{c^{1-\sigma}}{1-\sigma}-\alpha\frac{(y/\theta_i)^\gamma}{\gamma}$$

What are $h_C^i(C,L;\varphi)$ and $h_L^i(C,L;\varphi)$

Formulated as the static problem we had before we have and denoting by λ the Lagrange Multiplier associated to the resource constraint:

$$\varphi^i c_i^{-\sigma} = \lambda$$

$$\varphi^i (y_i/\theta_i)^{\gamma-1}\frac{1}{\theta_i} = \lambda$$

replacing these expressions in the resource constraints we can find that:

$$h_c^i(C,L;\varphi) = \omega_c^i C \quad \text{and} \quad h_y^i(C,Y;\varphi) = \omega_y^i L$$

with:

$$\omega_c^i = \frac{(\varphi^i)^{1/\sigma}}{\sum_i \pi_i(\varphi_i)^{1/\sigma}} \quad \text{and} \quad \omega_y^i = \frac{(\theta^i)^{\frac{\gamma}{\gamma-1}}(\varphi^i)^{\frac{-1}{\gamma-1}}}{\sum_i \pi_i(\theta^i)^{\frac{\gamma}{\gamma-1}}(\varphi^i)^{\frac{-1}{\gamma-1}}}$$

$$U^m = \Phi_u^m \frac{c^{1-\sigma}}{1-\sigma} - \Phi_v^m \alpha \frac{(y/\theta^i)^\gamma}{\gamma} \quad \text{and} \quad W = \Phi_u^W \frac{c^{1-\sigma}}{1-\sigma} - \Phi_v^W \alpha \frac{(y/\theta^i)^\gamma}{\gamma}$$

in which Φ_u^m, Φ_v^m, Φ_u^W and Φ_v^W are some constants.This will imply:

$$\tau^*(C,L) = 1 - \frac{\Phi_v^m}{\Phi_u^m}\frac{\Phi_u^W}{\Phi_v^W}$$

Furthermore:

$$\frac{U_C^m(C(s^t),L(s^t);\varphi)}{W_C(C(s^t),L(s^t);\varphi,\mu\lambda)}\frac{W_C(C(s^{t+1}),L(s^{t+1});\varphi,\mu\lambda)}{U_C^m(C(s^{t+1}),L(s^{t+1});\varphi)} = 1$$

so:

$$R(s^{t+1}) = R^*(s^{t+1})$$

and this implies that:

$$\kappa(s^t) = 0 \quad \forall t \geq 1$$

which means that the result of Chamley-Judd holds also in this environment for $t \geq 1$. Does this hold for every period? If $k_0^i = k_0 \forall i$ then taxing initial capital is equivalent to having a lump-sum tax. Since lump-sum taxes are not ruled out in this framework then taxing initial capital becomes unnecessary. With initial wealth heterogeneity and without available lump-sum taxes, initial wealth taxation is usually desirable.

Notice that this model nests the representative-agent that we studied originally (just by making $\theta^i = 1 \forall i$ and by ruling out lump-sum tax to zero. Why is the optimal capital income tax equal to zero? The optimal capital income tax is zero in this example because preferences are homothetic over consumption paths and separable (consumption and labor). Therefore, consumption at different dates should be taxed uniformly, which is equivalent to a zero capital income tax.

How about the labor income tax? First, notice that this tax rate is constant over time and across histories. However, although the tax rate remains constant across realizations of uncertainty, the stochastic processes governing government expenditure and technology does itself affect the level of this constant tax rate (tax rate is not necessarily constant across comparative statics). Why is it like that? Distortionary taxation acts like a redistribution mechanism: a positive tax rate makes the high-skilled workers pay more taxes than low-skilled workers. Optimal tax rate at any point of time balances distributional concerns against efficiency. The tax will be smooth because the determinants of inequality are constant over time and invariant to government expenditure or aggregate technology shocks. In other words, the marginal cost from distortions should be equal to the marginal benefit of redistribution. Since the latter is constant and invariant to policy, then the optimal distortionary tax is constant.

2.7.9 Exercise

Suppose:

$$U^i(c,y) = \alpha log(c) + (1-\alpha) log\left(1 - \frac{y}{\theta^i}\right)$$

- Find $h^i_C(C,L;\varphi)$, $h^i_Y(C,L;\varphi)$.
- Find $U^m(C,L;\varphi)$ and $W(C,L)$.
- Find $\tau^*(L)$ and $\kappa(s^t)$

2.8 Taxing Capital in Life Cycle Economies (Erosa and Gervais (2002))

Imagine an environment with individuals who live 2 periods (no mortality risk). Denote by t the period at which a generation is born. This means, in period t we have 2 generations living together: the generation that was born at t and the generation born at t-1. There is no population growth.

2.8.1 Endowments:

Each individual is endowed with one unit of time at each age j and can transform one unit of time into z_j units of efficient labor. z_j is interpreted as the labor productivity of an individual at age j. Individuals are also endowed with initial assets denoted by $a_{0,t}$. In general, we will denote individual variables by $x_{j,t}$, where j denotes the age of the individual and t the generation at which she was born.

2.8.2 Preferences:

Individuals derive utility from consumption and leisure. Preferences are represented by the following utility function:

$$U(c_{0,t}, l_{0,t}) + \beta U(c_{1,t}, l_{1,t})$$

2.8.3 Technology:

Production is done using capital and labor. They assume a production function with constant returns to scale, denoted by $f(k_t, l_t)$. In equilibrium then we will have:

$$r_t = f_k(k_t, l_t), \quad w_t = f_l(k_t, l_t),$$

2.8.4 Markets and Government:

Government wants to finance an exogenous stream of expenditures. Paper assumes the government has access to a set of fiscal instruments and to a commitment technology to implement its fiscal policy. The set of instruments is given by: government debt and proportional taxes on consumption, labor income, and capital income. Taxes are allowed to depend on the age of the individuals:

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t \left[\sum_{j=0,1} \tau^c_{t-j,j} c_{j,t-j} + \sum_{j=0,1} \tau^l_{t-j,j} w_t z_j l_{j,t-j} + \tau^k_{t-1,1}(r_t - \delta) a_{1,t-1} \right]$$

And the optimization problem of an individual is:

$$Max \quad U(c_{0,t}, l_{0,t}) + \beta U(c_{1,t}, l_{1,t})$$

$$s.t.$$

$$(1 - \tau^c_{0,t}) c_{0,t} + a_{1,t} \leq (1 - \tau^l_{0,t}) w_t z_0 l_{0,t}$$

$$(1 - \tau^c_{1,t}) c_{1,t} \leq (1 - \tau^l_{1,t}) w_t z_1 l_{1,t} + (1 + (1 - \tau^k_{1,t}))(r_t - \delta) a_{1,t}$$

What is the notion of the Planner's problem in this case? To have this notion, we first denote by U^t to the lifetime utility obtained by the generation born at period t given a sequence of consumption and leisure:

$$U^t = U(c_{0,t}, l_{0,t}) + \beta U(c_{1,t}, l_{1,t})$$

The government has a discount factor γ across generations. Them, the planner aims to maximize:

$$\sum_{t=0}^{\infty} \gamma^t U^t$$

Exercise: Show that the implementability constraint in this model is:

$$U_{c_{0,t}} c_{0,t} + U_{l_{0,t}} l_{0,t} + \beta (U_{c_{1,t}} c_{1,t} + U_{l_{1,t}} l_{1,t}) = 0 \tag{10}$$

Exercise: Show that a feasible allocation is implementable if and only if it satisfies (10). Something that you will need to think about is the fact that having age-dependent taxes is essential for this result to hold. Marginal rates of substitution between consumption and leisure are not necessarily constant across generations alive at any date, even for allocations that satisfy the implementability constraint.

2.8.5 Ramsey Problem:

The Ramsey problem solves the following:

$$max \sum_{t=0}^{\infty} \gamma^t [U(c_{0,t}, l_{0,t}) + \beta U(c_{1,t}, l_{1,t})]$$

$$s.t.$$

$$U_{c_{0,t}} c_{0,t} + U_{l_{0,t}} l_{0,t} + \beta \left(U_{c_{1,t}} c_{1,t} + U_{l_{1,t}} l_{1,t} \right) = 0; \quad (\gamma^t \lambda_t)$$

$$c_t + k_{t+1} = f(k_t, l_t) + (1 - \delta) k_t; \quad (\gamma^t \phi_t)$$

$$c_t = c_{0,t} + c_{1,t-1}$$

$$l_t = l_{0,t} + l_{1,t-1}$$

$$k_t = a_{1,t-1}$$

By taking first-order conditions to this problem we find:

$$\gamma^t U_{c_{0,t}} + \gamma^t \lambda_t \left(U_{c_{0,t}} + U_{cc_{0,t}} c_{0,t} + U_{lc_{0,t}} l_{0,t} \right) = \gamma^t \phi_t$$

$$\gamma^t \beta U_{c_{1,t}} + \gamma^t \beta \lambda_t \left(U_{c_{1,t}} + U_{cc_{1,t}} c_{1,t} + U_{lc_{1,t}} l_{1,t} \right) = \gamma^{t+1} \phi_{t+1}$$

$$\gamma^t U_{l_{0,t}} + \gamma^t \lambda_t \left(U_{l_{0,t}} + U_{ll_{0,t}} l_{0,t} + U_{lc_{0,t}} c_{0,t} \right) = \gamma^t \phi_t f_{lt}$$

$$\gamma^t \beta U_{l_{1,t}} + \gamma^t \beta \lambda_t \left(U_{l_{1,t}} + U_{ll_{1,t}} l_{1,t} + U_{cl_{1,t}} c_{1,t} \right) = \gamma^{t+1} \phi_{t+1} f_{lt+1}$$

$$\gamma^t \phi_t = \gamma^{t+1} \phi_{t+1} (1 - \delta + f_{kt+1})$$

From this system we obtain:

$$\frac{U_{c_{0,t}} + \lambda_t (U_{c_{0,t}} + U_{cc_{0,t}} c_{0,t} + U_{lc_{0,t}} l_{0,t})}{U_{c_{1,t}} + \lambda_t (U_{c_{1,t}} + U_{cc_{1,t}} c_{1,t} + U_{lc_{1,t}} l_{1,t})} == \beta (1 - \delta + f_{kt+1})$$

Steady State: The notion of steady state in this environment is the following: $(c_{0,t}, c_{1,t}, l_{0,t}, l_{1,t}, a_{1,t}) = (c_0, c_1, l_0, l_1, a_1)$ and $\lambda_t = \lambda$. Then, the solution of the Ramsey Problem satisfies the following condition in steady state:

$$\frac{U_{c_0} + \lambda (U_{c0} + U_{cc_0} c_0 + U_{l0} l_0)}{U_{c_1} + \lambda (U_{c1} + U_{cc_1} c_1 + U_{l1} l_1)} = \beta (1 - \delta + f_k)$$

2.8.6 Optimal Fiscal Policy

The solution to the Ramsey problem generally features nonzero taxes on labor and capital income. In contrast with infinitely-lived agent models, if the Ramsey allocation converges to a steady-state solution, optimal capital income taxes will generally be different from zero even in steady state. What is the intuition of this? Basically the difference comes from the fact that consumption and leisure are not constant in general over the life-cycle, in contrast with infinitely lived agent models, where this actually happens in the steady state. The Chamley-Judd result holds in the infinitely-lived economy regardless of the utility function. In this model, there are only very special cases in which the Chamley-Judd result holds.

Exercise: Assume the Utility Function is Additive Separable across time and that individuals discount the future at a fixed rate β. Also assume that $z_j = z \forall j$ and $\gamma = \beta$. Show that it is not optimal to tax capital income in the long-run with the set of fiscal instruments described above. What is the intuition?

Exercise: Assume the Utility function has the following form:

$$U(c, l) = V(G(c), l),$$

where $c = (c_0, c_1)$, $l = (l_0, l_1)$ and G(.) is homothetic. This means assuming the utility function is weakly separable. Show that the Ramsey problem prescribes zero taxes on capital income for time period 1 and thereafter provided labor income taxes can be age-conditioned. How do you relate this result to the Uniform Commodity Taxation result?

3 Static Mirrleesian Approach to Optimal Taxation

The Mirrleesian approach to Optimal Taxation is also known as the mechanism design approach of taxation. Something to notice in the Ramsey approach of optimal taxation is that we arbitrarily chose specific functional forms for the tax rates. In particular, so far we have been working with linear taxes (constant marginal rates) and we have excluded lump-sum taxes. Without these restrictions, lump sum taxes may become very attractive in these models. However, is this the best the government can do? what if the government can achieve better outcomes by using more sophisticated instruments (non-linear)? Equally important, what if there are some aspects of individuals that can not be observable by the government, like abilities? Is it really true that it is possible to implement desirable allocations if we ignore these features?

The Mirrleesian approach exploits information frictions/enforcement limitations of the government to pin down what are the instruments that the government needs to obtain desirable outcomes. In particular, the optimal policies are found among those that can deliver incentive-compatible allocations, given the information frictions or enforcement limitations. In other words, the government under this approach is careful to check whether a tax policy is implementable given the information limitations or not. How do we do this? Basically we have to follow two big steps:

1. Find a socially optimal allocation given the information limitations.

2. Design a tax system that can implement these allocations.

Before we jump to the canonical model, first let's introduce some terminology of mechanism design. If you are interested in learning more about mechanism design you can check any graduate level textbook of microeconomics.

3.1 Basic Terminology:

Suppose for simplicity that there are I agents, indexed by $i \in \{1, 2, ..., I\}$ who live $T < \infty$ periods. Suppose preferences are represented by the utility function:

$$\sum_{t=1}^{T} \beta^{t-1} \left(u(c_t) - v(l_t)\right)$$

with $0 < \beta < 1$ and $u(.), v(.)$ satisfying the usual assumptions. Notice that I am assuming additive separability in preferences. We will assume that while consumption c_t is observed, effort or hours worked l_t is only privately observed.

Denote by Θ the set of skills or ability. These skills are given to the individual. In other words, it is nature who draws $\theta_i^T = (\theta_i^1, ..., \theta_i^T) \in \Theta^T$ for each individual i. Assume θ_i^T is drawn i.i.d across agents. Denote by $\pi(.)$ the probability density function over Θ^T draws. What is the timeline in this environment? At the beginning of each period t, agents learn privately θ_{nt}. An individual with skill θ_{it} and works l_{it} hours produces y_{it} units of consumption good/output:

$$y_{it} = \theta_{it} l_{it}$$

It is going to be more convenient for us to work with y_{it} than with l_{it}. We assume that y_{it} is observable but both θ_{it} and l_{it} are not.

Definition 2 *An allocation is a sequence of functions* $(c_i, y_i)_{i=1}^{I}$*:*

$$c_i : (\Theta^T)^I \to \mathbb{R}_+^T$$

$$y_i : (\Theta^T)^I \to \mathbb{R}_+^T$$

such that c_{it} *and* y_{it} *are* $(\theta_1^t, ..., \theta_I^t)$ *- measurable.*

Definition 3 *An allocation* $(c_i, y_i)_{i=1}^{I}$ *is feasible if:*

$$\sum_{i=1}^{I} \sum_{t=1}^{T} R^{-t} c_{it}(\theta_1^T, ..., \theta_I^T) \leq \sum_{i=1}^{I} \sum_{t=1}^{T} R^{-t} y_{it}(\theta_1^T, ..., \theta_I^T), \quad R > 1$$

$\forall (\theta_1^T, ..., \theta_I^T)$ *such that* $\pi(\theta_1^T, ..., \theta_I^T)$*. Furthermore, denote by FA the set of feasible allocations.*

Now, is this enough to have guidance regarding policies? What is the set of implementable allocations given private information? The following definitions shed light about this. To make things simple, we will assume $T = 1$.

Definition 4 *A Mechanism is a set of actions* $(A_1, ..., A_I)$ *and outcome functions:*

$$(g^c, g^y) : \prod_{i=1}^{I} A_i \to FA$$

The timing is:

1. Nature draws θ for each individual i.
2. Agents privately observe θ_i and simultaneously choose an action $a_i \in A_i$.
3. Outcome is determined according to the outcome function.

Definition 5 *Let* (A, g^c, g^y) *be a Mechanism. a Bayesian Nash Equilibrium (BNE) is a collection of strategies* $\{\alpha_i^*\}_{i=1}^{I}$*,* $\alpha_i^* : \Theta \to A_n$ *such that:*

$$\alpha_i^*(\theta_i) \in \underset{a \in A_n}{argmax} \sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(g_i^c(\rho, \alpha_{-i}^*(\theta_{-i}))) - v\left(\frac{g_i^y(\rho, \alpha_{-i}^*(\theta_{-i})}{\theta_i} \right) \right)$$

We call $\{g_i^c(\alpha_1^*(\theta_1), ..., \alpha_I^*(\theta_I))\}_{i=1}^{I}$ *and* $\{g_i^y(\alpha_1^*(\theta_1), ..., \alpha_I^*(\theta_I))\}_{i=1}^{I}$ *an equilibrium outcome.*

Definition 6 *A feasible allocation* $(c_i, y_i))_{i=1}^I$ *is implementable if there is a mechanism* (A, g^c, g^y) *and a BNE* $\{\alpha_i^*\}_{i=1}^I$ *of such mechanism that satisfies:*

$$c_i = g_i^c(\alpha_1^*(\theta_1), ..., \alpha_I^*(\theta_I)), \quad y_i = g_i^y(\alpha_1^*(\theta_1), ..., \alpha_I^*(\theta_I))$$

The notion of implementability is pretty much clear. However, it sounds a bit abstract specially when we think of optimal fiscal policy. In particular, notice that this definition does not impose any restriction on the types of mechanisms considered. Optimal policy is about implementing the best possible allocation. Searching among different kinds of mechanisms (space of games) and finding the one that has a BNE that implements that best possible allocation sounds like a lot of work to do does not it? The good news is that there is a beautiful result that allow us to restrict our attention to a particular game without losing generality. Let me give you a couple of definitions to talk more about this:

Definition 7 *A **Direct** mechanism is such that* $A_i = \Theta \ \forall i$.

Definition 8 *A truth-telling BNE of a direct mechanism* (Θ, g^c, g^y) *is* $\alpha_i^*(\theta_i) = \theta_i \ \forall i$ *such that:*

$$\theta_i \in \underset{a \in \Theta}{ArgMax} \sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(g_i^c(a, \theta_{-i})) - v\left(\frac{g_i^y(a, \theta_{-i})}{\theta_i}\right)\right)$$

An allocation is truthfully implementable if:

$$c_i = g_i^c(\theta_1, ..., \theta_I), \quad y_i = g_i^y(\theta_1, ..., \theta_I),$$

In other words, in a direct mechanism, actions of individuals is the set of types. Individuals are asked to report their types (skills). We are interested in studying equilibria in which types are truthfully revealed. In the following result, we will see that we do not loss generality in doing that.

Proposition 7 *An allocation* $\{(c_i, y_i)\}_{i=1}^I$ *is implementable if and only if it is truthfully implementable in a direct mechanism.*

Proof. Only if part:

Suppose an allocation $(c_i, y_i)_{i=1}^I$ is implementable as outcome of some mechanism (A, g^c, g^y). We want to show that it is truthfully implementable in a direct mechanism. Let's construct such mechanism as follows:

$$\tilde{g}^c(\theta_1, ..., \theta_I) = g^c(\alpha_1^*(\theta_1), ..., \alpha_I^*(\theta_I)), \qquad \tilde{g}^y(\theta_1, ..., \theta_I) = g^y(\alpha_1^*(\theta_1), ..., \alpha_I^*(\theta_I)),$$

Where $\{\alpha_i\}_{i=1}^I$ is the BNE of some mechanism (A, g^c, g^y)). We have to show now that truth-telling is a BNE of $(\Theta, \tilde{g}^c, \tilde{g}^y)$. To prove this, suppose as a matter of a contradiction that there is a type θ_i and a report $\hat{\theta} \in \Theta$ such that:

$$\sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(\tilde{g}_i^c(\hat{\theta}, \theta_{-i})) - v \left(\frac{\tilde{g}_i^y(\hat{\theta}, \theta_{-i})}{\theta_i} \right) \right) > \sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(\tilde{g}_i^c(\theta_i, \theta_{-i})) - v \left(\frac{\tilde{g}_i^y(\hat{\theta}, \theta_{-i})}{\theta_i} \right) \right)$$

$$= \sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(g_i^c(\alpha_i^*(\theta_i), \alpha_{-i}^*(\theta_{-i}))) - v \left(\frac{g_i^y(\alpha_i^*(\theta_i)), \alpha_{-i}^*(\theta_{-i})}{\theta_i} \right) \right)$$

The last term follows from the definition of $(\tilde{g}^c, \tilde{g}^y)$. This implies that there must exist a $\beta = \alpha_n^{*-1}(\hat{\theta}) \in A_n$ such that:

$$\sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(g_i^c(\beta, \alpha_{-i}^*(\theta_{-i}))) - v \left(\frac{g_i^y(\alpha_i^*(\beta, \alpha_{-i}^*(\theta_{-i})}{\theta_i} \right) \right) >$$

$$\sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(g_i^c(\alpha_i^*(\theta_i), \alpha_{-i}^*(\theta_{-i}))) - v \left(\frac{g_i^y(\alpha_i^*(\theta_i)), \alpha_{-i}^*(\theta_{-i})}{\theta_i} \right) \right)$$

which is a contradiction of BNE. Therefore, $(c_i, y_i)_{i=1}^I$ can be implemented by:

$$c_i = \tilde{g}^c(\theta_1, ..., \theta_I), \quad y_i = \tilde{g}^y(\theta_1, ..., \theta_I),$$

If part: Obvious.

■ The Revelation Principle is so powerful. It allow us to restrict our attention to direct mechanisms and allocations that are truthfully revealing. This means that the set of implementable allocations are the ones satisfying the following condition:

$$\sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(c_i(\theta_i, \theta_{-i})) - v \left(\frac{y_i(\theta_i, \theta_{-i})}{\theta_i} \right) \right) \geq \sum_{\theta_{-i}} \pi(\theta_{-i}) \left(u(c_i(\hat{\theta}, \theta_{-i})) - v \left(\frac{y_i(\hat{\theta}, \theta_{-i})}{\theta_i} \right) \right)$$

$\forall i, \forall \hat{\theta} \in \Theta$. We are going to focus on studying environments in which there is a unit mass of agents.

3.2 The Canonical Model

Suppose $T = 1$ and that there are only two types θ_H and θ_L with $\theta_H > \theta_L$. I want to show you that whether θ is private information or not really matters in terms of implementation of allocations.

3.2.1 Public Information

Suppose the types of individuals is observed by everyone, in particular, by a utilitarian planner that wants to maximize:

$$\underset{c(\theta_L), c(\theta_H), y(\theta_L), y(\theta_H)}{Max} \quad \pi(\theta_H) \left[u(c(\theta_H)) - v \left(\frac{y(\theta_H)}{\theta_H} \right) \right] + \pi(\theta_L) \left[u(c(\theta_L)) - v \left(\frac{y(\theta_L)}{\theta_L} \right) \right]$$

$$s.t.$$

$$\pi(\theta_H)[c(\theta_H) - y(\theta_H)] + \pi(\theta_L)[c(\theta_L) - y(\theta_L)] = 0 \quad ...(\lambda)$$

This last constraint is the feasibility constraint. By taking first-order conditions we have:

$$u'(c(\theta_H)) = u'(c(\theta_L)) = \lambda \quad \Longrightarrow c(\theta_H) = c(\theta_L)$$

$$\frac{1}{\theta_H} v' \left(\frac{y(\theta_H}{\theta_H} \right) = \frac{1}{\theta_L} v' \left(\frac{y(\theta_L}{\theta_L} \right) = \lambda \Longrightarrow y(\theta_H) > y(\theta_L)$$

Therefore:

$$\underbrace{u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right)}_{\text{Pretending you are low type}} = u(c(\theta_H)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) > u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)$$

This means that the high-type has incentives to pretend that she is low-type. In other words, this is not incentive compatible and thus it can not be implemented as a BNE. Therefore, we can infer than when individuals have private information about their type, at least one incentive-compatibility constraint is going to be binding at the optimal solution.

3.2.2 Private Information

When we assume that the types are private information, the problem that the planner will have to solve is different. In particular, if we want to implement optimal allocation we will need to incorporate the incentive-compatibility constraints:

$$\underset{c_H, y_H, c_L, y_L}{Max} \quad \pi(\theta_H) \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right) \right] + \pi(\theta_L) \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right) \right]$$

$$s.t.$$

$$\pi(\theta_H)[c(\theta_H) - y(\theta_H)] + \pi(\theta_L)[c(\theta_L) - y(\theta_L)] = 0 \qquad (FEAS)$$

$$u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right) \geq u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \qquad (IC1)$$

$$u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right) \geq u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_L}\right) \qquad (IC2)$$

Something to notice about this problem is that it typically does not feature a concave objective function with a convex constraint set. Why? because $u(c)'s$ and $v's$ appear on both the left and the right hand side of the IC's constraints. Now, what are the properties of the allocations that satisfy both Feasibility and Incentive Compatibility? The following lemmas sumarize these properties:

Lemma 1 *If the contract (c_L, y_L) and (c_H, y_H) satisfies FEAS, IC1 and IC2, then one of the configurations must hold:*

- $c_H > c_L$ *and* $y_H > y_L$*;*
- $c_L > c_H$ *and* $y_L > y_H$*; or,*

- $c_L = c_H$ *and* $y_L = y_H$

Proof.

1. Suppose $c_H > c_L$ but $y_H \leq y_L$. If this is true, then:

$$u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_L}\right) > u(c(\theta_L)) - v\left(\frac{y(\theta_H)}{\theta_L}\right)$$

   since $c(\theta_H) > c(\theta_L)$ and u is monotone. Moreover:

$$u(c(\theta_L)) - v\left(\frac{y(\theta_H)}{\theta_L}\right) \geq u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)$$

   Then this violates IC2 and therefore these kind of allocations are not feasible.

2. A similar procedure is followed to show that combinations of $c_H \geq c_L$ and $y_H < y_L$ are also not feasible.

3. Now suppose $c_H < c_L$ but $y_H \geq y_L$. If this were true, then:

$$u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) > u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)$$

   IC1 would be violated.

4. Something similar follows to show that $c_H \leq c_L$ but $y_L < y_H$. So this can not be the case of the solution.

■ Now I will show that only one of those three possible configurations stated in Lemma 1 can be an optimal allocation. The following lemmas state this formally:

Lemma 2 *The configuration $c_L > c_H$ and $y_L > y_H$ is not feasible.*

Proof. Suppose as a matter of a contradiction that $c_L > c_H$ and $y_L > y_H$. By IC2 we would have:

$$U(c_L, y_L; \theta_L) - U(c_H, y_H; \theta_L) = u(c_L) - u(c_H) - \left[v\left(\frac{y_L}{\theta_L}\right) - v\left(\frac{y_H}{\theta_L}\right)\right] \geq 0$$

we can rewrite this as:

$$u(c_L) - u(c_H) - \frac{1}{\theta_L}\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_L}\right) dy \geq 0$$

Therefore:

$$\underbrace{u(c_L) - u(c_H)}_{>0 \quad (c_L > c_H)} \geq \frac{1}{\theta_L}\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_L}\right) dy$$

Since v(.) is convex and $y_L > y_H$, it follows that the second term is also positive. However, since $\theta_L < \theta_H$ and v(.) is convex, it follows that $v'\left(\frac{y}{\theta_H}\right) < v'\left(\frac{y}{\theta_L}\right) \forall y$, and as a result:

$$\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_H}\right) dy < \int_{y_H}^{y_L} v'\left(\frac{y}{\theta_L}\right) dy$$

Since $\theta_H > \theta_L$ we also have:

$$\frac{1}{\theta_H}\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_H}\right) dy < \frac{1}{\theta_L}\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_L}\right) dy$$

Then:

$$u(c_L) - u(c_H) \geq \frac{1}{\theta_L}\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_L}\right) dy > \frac{1}{\theta_H}\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_H}\right) dy$$

And notice that:

$$\frac{1}{\theta_H}\int_{y_H}^{y_L} v'\left(\frac{y}{\theta_H}\right) dy = v\left(\frac{y_L}{\theta_H}\right) - v\left(\frac{y_H}{\theta_H}\right)$$

Therefore:

$$u(c_L) - u(c_H) - \left[v\left(\frac{y_L}{\theta_H}\right) - v\left(\frac{y_H}{\theta_H}\right)\right] > 0$$

$$U(c_L, y_L; \theta_H) - U(c_H, y_H; \theta_H) > 0$$

which is a violation of IC1. ■

Lemma 3 *The configuration $c_L = c_H$ and $y_L = y_H$ is not optimal.*

Proof. To make this proof more intuitive, consider first an autarkic allocation in which each agent is left alone to consume whatever she produces. Formally, this means assuming that agents solve:

$$\underset{c,y}{Max} \quad u(c) - v\left(\frac{y}{\theta}\right)$$

$$s.t.$$

$$c \leq y$$

which can be rewritten as:

$$u(y) - v\left(\frac{y}{\theta}\right)$$

Taking FOC's we get:

$$u'(c(\theta)) = \frac{1}{\theta}v'\left(\frac{y(\theta)}{\theta}\right)$$

for each type. The higher is θ, the higher $c(\theta)$ has to be to hold this equation. As a result, higher types would be consuming and producing more in this autarkic setting. Notice also that this autarkic allocation satisfies Feasibility and both of the Incentive Compatibility Constraints. However, when $c_H = c_L$ and $y_H = y_L$, different types are again, in effect, in autarky (they consume whatever they produce). Now take a second to think: when the planner distorts the decisions by forcing individuals to produce differently than they would have done if left alone, would this result in an increase of average utility? Turns out that the answer is not and the intuition is that without providing any insurance or redistribution, this only has distortionary effects. Formally:

Suppose (c, y) denotes the consumption/production pair. Consider the following cases:

1. $u'(c) < \frac{1}{\theta_H} v'\left(\frac{y}{\theta_H}\right)$, Since $\theta_H > \theta_L$, then: $u'(c) < \frac{1}{\theta_L} v'\left(\frac{y}{\theta_L}\right)$. Therefore, by decreasing c and y at the same time by a small amount would keep the IC constraints holding and agents would be better off.

2. If $u'(c) = \frac{1}{\theta_H} v'\left(\frac{y}{\theta_H}\right)$, then $u'(c) < \frac{1}{\theta_L} v'\left(\frac{y}{\theta_L}\right)$. Therefore, decreasing production and consumption of low types would make them better off keeping high types without incentives to deviate to new allocation.

   I will leave you as an exercise to check what happens in the two following cases: 1) $u'(c) > \frac{1}{\theta_H} v'\left(\frac{y}{\theta_H}\right)$ and $u'(c) = \frac{1}{\theta_L} v'\left(\frac{y}{\theta_L}\right)$, and 2) $u'(c) > \frac{1}{\theta_H} v'\left(\frac{y}{\theta_H}\right)$ and $u'(c) < \frac{1}{\theta_L} v'\left(\frac{y}{\theta_L}\right)$.

■

Proposition 8 *At the optimal allocation, $c_H > c_L$ and $y_H > y_L$.*

Proposition 9 *At the optimal allocation, IC2 is not binding.*

Proof. Notice that we can rewrite the Planner's problem as follows:

$$\underset{c_L, y_L, y_L, y_H}{Max} \ \pi_H \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi_L \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right] \qquad (SP1)$$

$$s.t.$$

$$\pi_H c(\theta_H) + \pi_L c(\theta_L) \leq \pi_H y(\theta_H) + \pi_L y(\theta_L) \qquad (FEAS)$$

$$u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right) \geq u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \qquad (IC1)$$

$$u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right) \geq u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_L}\right) \qquad (IC2)$$

$$c(\theta_L) > c(\theta_H) \quad \text{and} \quad y(\theta_L) > y(\theta_H) \qquad (MONOT)$$

By the previous proposition, we know that the last constraint is redundant. Now consider a relaxed version of this problem:

$$\underset{c(\theta_L), y(\theta_L), y(\theta_L), y(\theta_H)}{Max} \ \pi_H \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi_L \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right] \qquad (RP1)$$

$$s.t.$$

$$\pi_H c(\theta_H) + \pi_L c(\theta_L) \leq \pi_H y(\theta_H) + \pi_L y(\theta_L) \qquad (FEAS)$$
$$u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right) \geq u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \qquad (IC1)$$

$$c(\theta_L) > c(\theta_H) \quad \text{and} \quad y(\theta_L) > y(\theta_H) \qquad (MONOT)$$

Since this problem has less constraints than the previous one, if the solution to (RP1) satisfies IC2, then it will be a solution for (SP1) as well. We will show this.

First, we show that (IC1) will be satisfied with equality in the solution of this problem. Suppose (c_L, y_L) and (c_H, y_H) are the solutions to this problem. Suppose as a matter of a contradiction that (IC1) is not satisfied with equality:

$$u(c_H) - v\left(\frac{y_H}{\theta_H}\right) > u(c_L) - v\left(\frac{y_L}{\theta_H}\right)$$

Then, because we are assuming u(.) satisfies usual assumptions (including continuity), I can keep this equality holding by adding a bit to (c_L) and subtracting a bit from c_H:

$$u(c_H - \epsilon) - v\left(\frac{y_H}{\theta_H}\right) > u(c_L + \delta) - v\left(\frac{y_L}{\theta_H}\right)$$

as long as ϵ and δ are small enough. Now, consider a new contract $(c_L + \delta, y_L)$ and $(c_H - \epsilon, y_H)$ and choose $\delta = \frac{\pi_H \epsilon}{\pi_L}$. Then, if we make ϵ small enough, we have that IC1 holds and Feasibility will be satisfied:

$$\pi_H(c_H - \epsilon) + \pi_L(c_L + \epsilon) = \pi_H c_H + \pi_L c_L + \pi_H \epsilon - \pi_L \frac{\pi_H}{\pi_L}\epsilon = \pi_H c_H + \pi_L c_L$$

So this contract is feasible. Now, we show that this contract induces an increase in welfare even when ϵ is small but positive. Let's compute that change in welfare:

$$\Delta W = \pi_H \left[u(c_H - \epsilon) - u(c_H)\right] - \pi_L \left[u\left((c_L + \frac{\pi_H \epsilon}{\pi_L}\right) - u(c_L)\right]$$

Now, let's take the derivative of this change with respect to ϵ, at $\epsilon = 0$:

$$\frac{\partial \Delta W}{\partial \epsilon} = -\pi_H u'(c_H) + \pi_L \frac{\pi_H}{\pi_L} u'(c_L),$$
$$= -\pi_H u'(c_H) + \pi_H u'(c_L) = \pi_H[u'(c_L) - u'(c_H)] > 0,$$

this last inequality follows from the concavity of u(.). Thus, we can rewrite our problem as:

$$\underset{c(\theta_L), y(\theta_L), y(\theta_L), y(\theta_H)}{Max} \pi_H \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi_L \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right] \qquad (RP2)$$

$$s.t.$$

$$\pi_H c(\theta_H) + \pi_L c(\theta_L) \leq \pi_H y(\theta_H) + \pi_L y(\theta_L) \qquad (FEAS)$$
$$u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right) = u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \qquad (IC1)$$
$$c(\theta_L) > c(\theta_H) \quad \text{and} \quad y(\theta_L) > y(\theta_H) \qquad (MONOT)$$

To finish showing that (IC2) is redundant, it is sufficient to show that at the solution to (RP1), (IC2) is satisfied. In other words, we need to show that if:

$$u(c_H) - v\left(\frac{y_H}{\theta_H}\right) = u(c_L) - v\left(\frac{y_L}{\theta_H}\right),$$

and

$$c_L < c_H \quad \text{and} \quad y_H < y_L$$

then

$$u(c_L) - v\left(\frac{y_L}{\theta_L}\right) > u(c_H) - v\left(\frac{y_H}{\theta_L}\right)$$

The proof is very easy:

$$v\left(\frac{y_H}{\theta_L}\right) - v\left(\frac{y_L}{\theta_L}\right) = \int_{y_L}^{y_H} v'\left(\frac{y}{\theta_L}\right) dy > \int_{y_L}^{y_H} v'\left(\frac{y}{\theta_H}\right) dy = v\left(\frac{y_H}{\theta_H}\right) - v\left(\frac{y_L}{\theta_H}\right)$$
$$= u(c_H) - u(c_L)$$

Therefore:

$$u(c_L) - v\left(\frac{y_L}{\theta_L}\right) > u(c_H) - v\left(\frac{y_H}{\theta_L}\right)$$

Another way to see this is by taking SP1 directly, let's take FOCs in the SP1. For this purpose, denote by λ, $\mu(\theta_H, \theta_L)$, and $\mu(\theta_L, \theta_H)$ to the Lagrange Multipliers on the resource constraint, IC1 and IC2, respectively:

$$c_L: \quad \pi_L u'(c_L) = \lambda \pi_L + \mu(\theta_H, \theta_L) u'(c_L) - \mu(\theta_L, \theta_H) u'(c_L), \tag{11}$$

$$c_H: \quad \pi_L u'(c_L) = \lambda \pi_H - \mu(\theta_H, \theta_L) u'(c_H) + \mu(\theta_L, \theta_H) u'(c_H), \tag{12}$$

$$\pi_L \frac{1}{\theta_L} v'\left(\frac{y_L}{\theta_L}\right) = \lambda \pi_L + \mu(\theta_H, \theta_L) \frac{1}{\theta_H} v'\left(\frac{y_L}{\theta_H}\right) - \mu(\theta_L, \theta_H) \frac{1}{\theta_L} v'\left(\frac{y_L}{\theta_L}\right); \quad \text{and} \tag{13}$$

$$\pi_H \frac{1}{\theta_H} v'\left(\frac{y_H}{\theta_H}\right) = \lambda \pi_H - \mu(\theta_H, \theta_L) \frac{1}{\theta_H} v'\left(\frac{y_H}{\theta_H}\right) + \mu(\theta_L, \theta_H) \frac{1}{\theta_L} v'\left(\frac{y_H}{\theta_L}\right); \tag{14}$$

Something that you have to keep in mind is that these are not sufficient conditions because this problem is not a concave programming one (because of the potential non-convexity of the feasible set). Also note that in showing that $c_H = c_L$ and $y_H = y_L$ cannot be a solution to (SP1) we did not use any of the previous steps. Try showing now that

(IC1) holds with equality (should not be hard to do it!). These two results will imply that IC2 will be slack in the solution. This will imply $\mu(\theta_H, \theta_L) > 0$ and $\mu(\theta_L, \theta_H) = 0$ in the optimum:

$$\left[1 + \frac{\mu(\theta_L, \theta_H) - \mu(\theta_H, \theta_L)}{\pi_L}\right] u'(c_L) = \lambda,$$

$$\left[1 + \frac{\mu(\theta_H, \theta_L) - \mu(\theta_L, \theta_H)}{\pi_H}\right] u'(c_H) = \lambda,$$

Since $\mu(\theta_H, \theta_L)$ is a LM it must be true that:

$$\frac{\mu(\theta_H, \theta_L)}{\pi_L} > 0$$

Also, since u(.) is strictly increasing and λ is also a LM, then:

$$1 - \frac{\mu(\theta_H, \theta_L)}{\pi_L} \geq 0$$

$$0 < 1 - \frac{\mu(\theta_H, \theta_L)}{\pi_L} \leq 1$$

This implies $c_H > c_L$ in the optimum. Also, we know that at the solution:

$$u(c_H) - v\left(\frac{y_H}{\theta_H}\right) = u(c_L) - v\left(\frac{y_L}{\theta_H}\right)$$

when $c_H > c_L$ the only possibility for this last expression to hold is that $y_L < y_H$. ■ The following proposition summarizes all of these results.

Proposition 10 *The solution to (RP2) is the same as the solution of (SP1). The optimal allocation is monotone in type, (IC1) holds with equality and (IC2) is slack.*

Exercise: Write down this same framework for an arbitrary number of types (N) and show how these results look like in that case. **Hint:** should be similar that the 2-types case.

3.2.3 Characterization and Implementation of the Solution

A first-question you must be asking yourself right now is: Who is subsidizing who? In other words, we know that the high-type consumes and produces more than the low-type. But, is he consuming more than what he produces? or not? To get this answer, we need to get a result concerning marginal tax rates. Using the last proposition we got, we can write the planning problem as:

$$\underset{c(\theta_L), y(\theta_L), y(\theta_L), y(\theta_H)}{Max} \pi_H \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi_L \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right] \qquad (SP2)$$

$$s.t.$$

$$\pi_H c(\theta_H) + \pi_L c(\theta_L) = \pi_H l(\theta_H)\theta_H + \pi_L l(\theta_L)\theta_L \qquad (FEAS)$$

$$u(c(\theta_H)) - v\left(l(\theta_H)\right) = u(c(\theta_L)) - v\left(\frac{\theta_L}{\theta_H} l(\theta_L)\right) \qquad (IC)$$

Denote by λ and μ the LM of the feasibility constraint and the IC constraint, respectively. Then, the Lagrangian can be written as:

$$\mathcal{L} = \pi_H \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi_L \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right] + \lambda\left[\pi_H l(\theta_H)\theta_H + \pi_L l(\theta_L)\theta_L - \pi_H c(\theta_H) - \pi_L c(\theta_L)\right]$$

$$+\mu\left[u(c(\theta_H)) - v\left(l(\theta_H)\right) - u(c(\theta_L)) + v\left(\frac{\theta_L}{\theta_H} l(\theta_L)\right)\right]$$

The FOC's are:

$$c_H: \quad \pi_H u'(c_H) + \mu u'(c_H) = \pi_H \lambda;$$

$$c_L: \quad \pi_L u'(c_L) + \mu u'(c_L) = \pi_L \lambda;$$

$$l_H: \quad \pi_H v'(l_H) + \mu v'(l_H) = \pi_H \theta_H \lambda;$$

$$l_L: \quad \pi_L v'(l_L) + \mu \frac{\theta_L}{\theta_H} v'\left(\frac{\theta_L}{\theta_H} l_L\right) = \pi_L \theta_L \lambda;$$

Doing some algebra we have:

$$\frac{(\pi_H + \mu)v'(l_H)}{(\pi_H + \mu)u'(c_H)} = \frac{\pi_H \theta_H \lambda}{\pi_H \lambda}$$

which means

$$\frac{v'(l_H)}{\theta_H} = u'(c_H) \tag{15}$$

which is exactly the same condition that we obtained in the Full-Information case. We also have from this FOCs that:

$$\frac{\pi_L v'(l_L) - \mu \frac{\theta_L}{\theta_H} v'\left(\frac{\theta_L}{\theta_H} l_L\right)}{(\pi_L - \mu)u'(c_L)} = \frac{\pi_L \theta_L \lambda}{\pi_L \lambda},$$

Then:

$$\underbrace{\frac{\pi_L - \mu \frac{\theta_L}{\theta_H} v'\left(\frac{\theta_L}{\theta_H} l_L\right)/v'(l_L)}{(\pi_L - \mu)}}_{distortion} v'(l_L) = \theta_L u'(c_L) \tag{16}$$

Now I will show you that $\mu < \pi_L$. To see this, first notice that from the FOCs we have:

$$\frac{(\pi_H + \mu)u'(c_H)}{(\pi_L - \mu)u'(c_L)} = \frac{\pi_H}{\pi_L},$$

so

$$(\pi_H + \mu)u'(c_H) = \frac{\pi_H}{\pi_L}(\pi_L - \mu)u'(c_L)$$

Since $(\pi_H + \mu)u'(c_H) > 0$, $\frac{\pi_H}{\pi_L}u'(c_L) > 0$, it follows that $\pi_L - \mu > 0$. Now, since $v'(.)$ is strictly increasing and $\theta_L < \theta_H$, we have:

$$v'\left(\frac{\theta_L}{\theta_H} l_L\right) < v'(l_L),$$

which implies $v'\left(\frac{\theta_L}{\theta_H} l_L\right) / v'(l_L) < 1$, and, hence:

$$\frac{\theta_L}{\theta_H} v'\left(\frac{\theta_L}{\theta_H} l_L\right) < 1$$

$$\frac{\theta_L}{\theta_H} v'\left(\frac{\theta_L}{\theta_H} l_L\right) \mu < \mu$$

And therefore:

$$\frac{\pi_L - \mu \frac{\theta_L}{\theta_H} \frac{v'\left(\frac{\theta_L}{\theta_H} l_L\right)}{v'(l_L)}}{(\pi_L - \mu)} > 1$$

because $\mu < \pi_L$. From (16), we have:

$$v'(l_L) = \frac{(\pi_L - \mu)}{\pi_L - \mu \frac{\theta_L}{\theta_H} \frac{v'\left(\frac{\theta_L}{\theta_H} l_L\right)}{v'(l_L)}} u'(c_L) < \theta_L u'(c_L)$$

$$\frac{v'(l_L)}{\theta_L} < u'(c_L) \tag{17}$$

Meaning that the value of leisure of low-type individuals is less than their productivity at the margin. In other words, there is AN IMPLICIT TAX τ such that:

$$\frac{v'(l_L)}{(1 - \tau)\theta_L} = u'(c_L)$$

we must have that $0 < \tau < 1$. Equations (15) and (17) indicate the result of this literature called **No distortion at the top.** Meaning that the high-types are undistorted at the margin, but the low-types are.

3.2.4 Average Tax Rates

In this section we will see that average taxes are negative for the low type and positive for the high type.

Lemma 4 *In the optimal allocation:*

$$y_H - c_H > 0 > y_L - c_L$$

Proof. We know that at the optimal allocation, (IC1) holds with equality:

$$u(c_H) - u(c_L) = v\left(\frac{y_H}{\theta_H}\right) - v\left(\frac{y_L}{\theta_H}\right)$$

which can be written as:

$$\int_{c_L}^{c_H} u'(c)dc = \int_{y_L}^{y_H} \frac{1}{\theta_H} v'\left(\frac{y}{\theta_H}\right) dy$$

We also know that:

$$u'(c_H)(c_H - c_L) < \int_{c_L}^{c_H} u'(c)dc < u'(c_L)(c_H - c_L)$$

Because u(.) is a continuous function, by the Intermediate Value Theorem we know $\exists c^*$ such that:

$$\int_{c_L}^{c_H} u'(c)dc = u'(c^*)(c_H - c_L) \quad \text{for some } c^* \in (c_L, c_H)$$

Analogously:

$$\int_{y_L}^{y_H} \frac{1}{\theta_H} v'\left(\frac{y}{\theta_H}\right) dy = (y_H - y_L)\frac{1}{\theta_H} v'\left(\frac{y^*}{\theta_H}\right) \quad \text{for some } y^* \in (y_L, y_H)$$

Then:

$$(y_H - y_L)\frac{1}{\theta_H} v'\left(\frac{y^*}{\theta_H}\right) = u'(c^*)(c_H - c_L)$$

Now let's exploit the concavity of u(.) and the convexity of v(.):

$$u'(c^*) > u'(c_H) \text{ and } \frac{1}{\theta_H} v'\left(\frac{y^*}{\theta_H}\right) < \frac{1}{\theta_H} v'\left(\frac{y_H}{\theta_H}\right)$$

so:

$$u'(c^*) > u'(c_H) = \frac{1}{\theta_H} v'\left(\frac{y_H}{\theta_H}\right) > \frac{1}{\theta_H} v'\left(\frac{y^*}{\theta_H}\right)$$

Thus:

$$(c_H - c_L) < (y_H - y_L)$$

Meaning that:

$$y_H - c_H > y_L - c_L$$

From Feasibility constraint we must have that:

$$y_H - c_H > 0 > y_L - c_L \quad \text{i.e.,} \quad y_H > c_H \quad \text{and} \quad y_L < c_L$$

■

3.2.5 Implementation with Optimal Taxes

Let's think now on how we would implement this optimal allocations through decentralized decisions by workers subject to income taxes. In other words, we want to find a tax scheme $T(y)$ for each type in our economy. The contractual allocation (c, y) is going to be the solution of the following problem:

$$\underset{c,y}{Max} \quad u(c) - v\left(\frac{y}{\theta_i}\right)$$

$$s.t.$$

$$c \leq y - T(y)$$

Something that I need to remark is that there is more than one way to implement this optimal allocations. For example, you could think of the very trivial tax-scheme in which $T(y_L) = y_L - c_L$, $T(y_H) = y_H - c_H$ and $T(y) = y$ for any other different value. Another option (a more interesting one) is choosing T(y) so the mapping $y \to y - T(y)$ follows the indifference curve of the low-type for y's below y_L, and follows that of the high-type for y's above y_L; i.e.

$$\text{for } y \leq y_L, \quad u(y - T(y)) - v\left(\frac{y}{\theta_L}\right) = u(y_L - T(y_L)) - v\left(\frac{y_L}{\theta_L}\right); \quad \text{and,}$$
$$\text{for } y \geq y_L, \quad u(y - T(y)) - v\left(\frac{y}{\theta_H}\right) = u(y_H - T(y_H)) - v\left(\frac{y_H}{\theta_H}\right);$$

This schedule would have the following effect: It will make the low-type agent indifferent between picking any $y \leq y_L$, and the high type indifferent between picking any $y \geq y_L$. In addition, given the characterization of the contract above, the low-type will be strictly worse off by picking any $y > y_L$, and the high type strictly worse by picking $y < y_L$.

Just for fun, let's assume that the effective tax schedule for type i takes the form $T_i(y_i) = a_i + \tau_i y_i$, at least near the optimal contractual solution y_i. Let's check the properties of this tax schedule!

Consider first the FOCs of the household problem and denote by λ the LM of the budget constraint:

$$c: \quad u'(c) = \lambda; \quad \text{and;}$$
$$y: \quad \frac{1}{\theta}v'\left(\frac{y}{\theta}\right) = \lambda[1 - T'(y)],$$

we get the following intratemporal condition:

$$\frac{1}{\theta}v'(y\theta) = u'(c)[1 - T'(y)], \tag{18}$$

If we match equation (15) with equation (18) we corroborate that:

$$T'_H(y_{\theta_H}) = 0,$$

which implies:

$$\tau_H = 0$$

meaning that the tax schedule for the high types must look like a lump-sum tax at the optimal level of output (no distortion at the top of the type distribution).

Now let's match equation (17) with equation (18). As we can see, we get that:

$$0 < \tau_L < 1$$

meaning that the margin must be distorted for low types. Notice that the distortion is towards less work and more leisure. Why would the planner like the low-type to work less? let's characterize further our optimal tax schedule: So far we know that the Tax schedule should look like $T(y) = a_\theta + \tau_\theta y$, with $\tau_H = 0$ and $\tau_L \in (0, 1)$. What can be said about a_θ for each type?

Since the high types consumes less than his output and faces zero marginal tax rate, we should have that $a_H > 0$ near y_H. Then, for the resulting allocation to be optimal, feasibility must hold with equality at the optimum, implying that $a_L < 0$ near y_L.

In summary, we would have that the tax schedule looks exactly like lump-sum near (c_H, y_H), whereas, near (c_L, y_L) it looks like a linear tax. This can be written as:

$$T^*(y) = -T_L + T'(y_L)y_L \quad \forall y \leq y_L, \quad \text{and}$$

$$T^*(y) = T_H, \quad \forall \quad y \geq y_L \quad \text{(a lump sum tax)}$$

with T_L, T_H, and $T'(y_L)$ strictly positive. The expression for $\tau_L = T'(y_L)$ was given above. To find T_L and T_H we can use the budget constraints to get:

$$T_L = c_L - (1 - \tau_L)y_L > 0 \quad \text{because} \quad c_L > y_L > (1 - \tau_L)y_L$$

and

$$T_H = y_H - c_H > 0$$

so why does the planner want to tax output for the low-types, creating a distortion in output? subsidizing the output of low types creates an incentive for the high-types to pretend they are of the low-type. To prevent this happening, the planner must keep the low-types output-consumption schedule on the indifference curve of the high-types. This is done by putting a linear tax on income near y_L. This linear tax avoids high-types to be lying.

3.2.6 Labor Supply Implications

We already know that in the optimum $y_H > y_L$. But who is working more? Is it true that $l_H > l_L$? This is not easy to answer ... my Professors say that there is very little know about this. However, to have an idea of what is going on, we will use 2 elements 1) what we know about the marginal tax rate for the low-types and 2) the fact that there is a transfer from the high-type to the low-type.

Suppose that there tax scheme looks like the one described in the previous section: a two-part tax schedule that is linear near the optimal contract. Define $l((1-\tau)w, T)$ as the solution to:

$$\underset{c,l}{Max} \quad u(c) - v(l)$$

$$s.t.$$

$$c \leq w(1-\tau)l + T$$

Notice first that leisure is a normal good in this problem, and therefore $l((1-\tau)w, T)$ is strictly decreasing in T, keeping τ and w constant. Then:

$$l_H = y_H \theta_H = l((1-\tau)w = \theta_H, T_H < 0) > l(\theta_H, 0);$$

meaning that since the high-type suffers a negative transfer and has a zero marginal tax rate, he works more than he would without taxes (in autarky).

Furthermore, we know that labor supply has an upward slope in net wages: $l((1-\tau)w, T)$ is increasing in $(1-\tau)w$, holding T fixed. Under this assumption:

$$l_L = l\left([1 - T'(y_L)]\theta_L, T_L\right) < l(\theta_L, T_L) < l(\theta_H, T_L) < l(\theta_H, 0),$$

The last step comes from the fact that $T_L > 0$, given that leisure is a normal good. As a result we have:

$$l_L < l(\theta_H, 0) < l(\theta_H, T_H) = l_H$$

Since leisure is a normal good, and since $T_H < 0$, $l_H > l_H^{ce}$, ce denotes competitive equilibrium or in this case, the autarky situation.

Consider a θ-type consumer. Under the linear tax system:

$$\underset{c,l}{Max} \quad u(c) - v(l)$$

$$s.t.$$

$$c \leq (1-\tau)wl + T$$

from FOCs:

$$(1-\tau)u'(c) = \frac{1}{\theta}v'(l),$$

In autarky or the ex-post CE, where $\tau = T = 0$ for all types, this condition becomes:

$$u'(c^{ce}) = 1\theta v'(l^{ce}),$$

But when $\tau \in (0,1)$, in particular for the low-types we will have:

$$\frac{1}{\theta_L}v'(l_L) < u'(c_L)$$

We already know that the consumption of the low-types is higher under the optimal contract, relative to the ex-post competitive equilibrium; $c_L > c_L^{ce}$. Then:

$$u'(c_L) < u'(c_L^{ce}),$$

because of the concavity of u(.). Therefore:

$$u'(c_L) < u'(c_L^{ce}) = \frac{1}{\theta_L}v'(l_L^{ce}),$$

implying that:

$$\frac{1}{\theta_L}v'(l_L) < \frac{1}{\theta_L}v'(l_L^{ce})$$

meaning that $l_L < l_L^{ce}$ so low types work less under the optimal contract than in the competitive equilibrium. Therefore, the relationship between l_L and l_H could be found if we knew the relationship between the ex post competitive equilibrium labor supplies, l_L^{ce} and l_H^{ce}. If $l_L^{ce} \leq l_H^{ce}$ we can easily conclude that $l_L < l_H$.

This is the significance of assuming a labor supply that is upward sloping in net wages.

Exercise: Suppose preferences are given by:

$$u(c,l) = \alpha log(c) + (1-\alpha)log(1-l)$$

Compute T_H and T_L, l_H, l_L, l_H^{ce}, l_L^{ce}. What is the relation between l_L^{ce} and l_H^{ce}.

4 The New Dynamic Public Finance

So far we have characterized the set of achievable allocations by any mechanism. The goal of the planner is now to find the best achievable allocation in:

$$\sum_{t=1}^{T}\beta^{t-1}\sum_{\theta^T\in D}\pi(\theta^T)\omega(\theta_1)\left[u(c_t(\theta^T))-v\left(\frac{(y_t(\theta^T))}{\theta_t}\right)\right]$$

$$s.t$$

$$\sum_{t=1}^{T}\sum_{\theta^T}\left[u(c_t(\theta^T))-v\left(\frac{y_t(\theta^T))}{\theta_t}\right)\right]\geq\sum_{t=1}^{T}\sum_{\theta^T}\left[u(c_t(\alpha_t'(\theta^T))-v\left(\frac{y_t(\alpha_t'(\theta^T))}{\theta_t}\right)\right]$$

for all $\alpha' : D \to D$ (α_t' is θ^t-measurable).

$$\sum_{\Theta\in D}\sum_{t=1}^{T}c_t(\theta^T)\pi(\theta^T)/R^{t-1}\leq\sum_{\Theta\in D}\sum_{t=1}^{T}y_t(\theta^T)\pi(\theta^T)/R^{t-1}$$

$$(c_t, y_t) \text{ are } \theta^T - \text{measurable}$$

$$c_t(\theta^T), y_t(\theta^T)\geq 0 \quad \forall t, \theta^T\in D$$

Notice that we are allowing for the allocation to depend on the realization of θ. In this chapter, our goal is to characterize the properties of constraint efficient allocations (i.e. allocations that solve the above planner's problem). More interestingly, we want to identify what are the inter-temporal distortions that arise from solving this problem.

As we did in the previous chapter, first we study the properties of the model under full information.

4.1 Full information optima

Suppose θ_t is public information. Then, the planning problem is exactly the same but without the incentive-compatibility constraint. Denote by λ the multiplier on the feasibility constraint. Then:

$$\sum_{\theta^T}\pi(\theta^T)\omega(\theta_1)u'(c_t(\theta^T))\beta^{t-1}=\lambda R^{t-1}\sum_{\theta^T}\pi(\theta^T)$$

The equation above is the FOC with respect to $c_t(\theta^T)$ at a particular draw θ^T. We know that $c_t(\theta^T)$ is $\theta^t - measurable$, therefore we don't need to sum over all $\theta^T \in D$, but only those that contain the particular history θ^t.

$$\sum_{\theta^T\backslash\theta^t}\pi(\theta^T)\omega(\theta_1)u'(c_t(\theta^T))\beta^{t-1}=\frac{\lambda}{R^{t-1}}\sum_{\theta^T\backslash\theta^t}\pi(\theta^T)$$

by measurability of $c_t(\theta^T)$.

$$u'(c_t(\theta^T))\beta^{t-1}\omega(\theta_1)=\lambda/R^{t-1}$$

Note: his implies that $c_t(\theta^T)$ is θ_1-measurable at the optimum, meaninig that it is independent from θ_t fpr $t > 1$. In other words, the planner will provide full insurance. Furthermore, we can obtain the following Euler Equation:

$$u'(c_t(\theta^T)) = \beta R \mathbb{E}\left[u'(c_{t+1}(\theta^T)) \backslash \theta^t\right]$$

planner is happy to allow access to outside trade. Another thing to notice is that the following Euler equation also holds:

$$\frac{1}{u'(c_t(\theta^T))} = \lambda^{-1}\beta^{t-1}R^{t-1}\omega(\theta_1) = \frac{1}{\beta R}\mathbb{E}\left[\frac{1}{u'(c_{t+1}(\theta^T)}\backslash \theta^t\right]$$

4.1.1 Inverse Euler Equation

Consider again the planner problem with incentive-compatibility constraints. Denote by (c^*, y^*) the solution to this problem. Now, consider the following perturbation around (c^*, y^*):

$$y' = y^*$$

$$c'_s = c^*_s \quad \forall \quad s \neq t, t+1 \quad \text{(for fixed t)}$$

for all histories θ^t

$$u(c'_t(\theta^T)) + \beta u(c'_{t+1}(\theta^T)) = k + u(c^*_t(\theta^T)) + \beta u(c^*_{t+1}(\theta^T)) \quad \forall \theta^{t+1} \quad \text{such that} \quad \pi(\theta^{t+1} \backslash \theta^t) > 0$$

$$\sum_{\theta^T \backslash \theta^t} \pi(\theta^T)[c'_t(\theta^T) + c'_{t+1}(\theta^T)/R] = \sum_{\theta^T \backslash \theta^t} \pi(\theta^T)[c^*_t(\theta^T) + c^*_{t+1}(\theta^T)/R]$$

Something to notice is that (c', y') is feasible and incentive compatible. What exactly are we doing here?

We are perturbing $u(c_t(\theta^T))$ by some amount and then make an appropriate perturbation in every immediate history following θ^t so that incentive compatibility is preserved. If (c^*, y^*) is the solution to the Planner's problem, this perturbation cannot improve welfare. One implication of this is that (c^*, y^*) solves the following maximization problem and $k = 0$ at the optimal solution:

$$\underset{k, c'_t(\theta^T), c'_{t+1}(\theta^T)}{Max} \quad k$$

$$s.t.$$

$$u(c'_t(\theta^T)) + \beta u(c'_{t+1}(\theta^T)) = k + u(c^*_t(\theta^T)) + \beta u(c^*_{t+1}(\theta^T)) \quad \forall \quad \theta^t, \theta^{t+1}$$

such that $\pi(\theta^{t+1}) \backslash \theta^t) > 0$

$$\sum_{\theta^T \backslash \theta^t} \pi(\theta^T)[c'_t(\theta^T) + c'_{t+1}(\theta^T)/R] = \sum_{\theta^T \backslash \theta^t} \pi(\theta^T)[c^*_t(\theta^T) + c^*_{t+1}(\theta^T)/R]$$

Denote by $\eta(\theta^{t+1})$ and λ the Lagrange Multipliers associated to this problem. If we take FOCs we get:

$$\sum_{\theta^{t+1}\backslash\theta^t} \eta(\theta^{t+1})u'(c'_t(\theta^T)) = \lambda \sum_{\theta^T\backslash\theta^t} \pi(\theta^T) = \lambda\pi(\theta^t)$$

for all θ^t, θ^{t+1} such that $\pi(\theta^{t+1}\backslash\theta^t) > 0$.

$$\beta u'(c'_t(\theta^{t+1}))\eta(\theta^{t+1}) = \frac{\lambda}{R}\sum_{\theta^T\backslash\theta^{t+1}} \pi(\theta^T) = \lambda\pi(\theta^{t+1})/R$$

Substitute for $\eta(\theta^{t+1})$:

$$\sum_{\theta^{t+1}\backslash\theta^t} \frac{\lambda\pi(\theta^{t+1})/R}{\beta u'(c'_t(\theta^{t+1}))} u'(c'_t(\theta^T)) = \lambda\pi(\theta^t)$$

Cancel terms adn evaluate this at the solution $c' = c^*$:

$$\frac{\beta R}{u'(c^*_t(\theta^T))} = \sum_{\theta^{t+1}\backslash\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} \frac{1}{u'(c^*_{t+1}(\theta^T))} \tag{19}$$

Something important here is that the intertemporal condition only depends on consumption, which is observable. Something you have to notice here is that to derive this result, a key assumption was having additive separability in the utility function. This result implies that it is not desirable for the planner allowing access to savings. Why? to see why let's look at the following Euler equation (which holds if there is access to savings):

$$u'(c_t(\theta^t)) = \beta R \sum_{\theta^{t+1}\backslash\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t} u'(c_{t+1}(\theta^T)) \tag{20}$$

How is this related to our Inverse Euler Equation?

$$\begin{aligned} u'(c_t(\theta^T)) &= \beta R \frac{1}{\sum_{\theta^{t+1}\backslash\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} \frac{1}{u'(c^*_{t+1}(\theta^T))}} \\ &> \beta R \frac{1}{\frac{1}{\sum_{\theta^{t+1}\backslash\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c^*_{t+1}(\theta^T))}} \end{aligned}$$

so notice that in a 'constrained-efficient' allocation, individuals are 'constrained-efficient'. This means, if individuals can 'privately save'. they will choose to do so and it is desirable for the planner to prevent them to do that. Another way of seeing this is by the following:

Suppose equation (20) holds. Then we must have:

$$\frac{\beta R}{u'(c^*_t(\theta^T))} < \sum_{\theta^{t+1}\backslash\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} \frac{1}{u'(c^*_{t+1}(\theta^T))}$$

Suppose now that the planner wants to increase utility at time t by ϵ and decrease it at time $t+1$ by $\beta^{-1}\epsilon$. The cost of increase of utility in period 1 is $u'(c_t(\theta^T))/\epsilon$. At the same time, the planner hands in $u'(c_{t+1}(\theta^T))/\epsilon$ less at each θ^{t+1} that follows θ^t. Therefore it can free up resources.

4.1.2 On dynamics of consumption

Consider again the full information optimal allocation:

$$u'(c_t(\theta^T))\beta^{t-1}\omega(\theta_1) = \frac{\lambda}{R^{t-1}}$$

Assume for simplicity $\beta R = 1$. Then:

1. Allocation is independent of history.

2. There is no mobility in short-run or long-run.

3. Inequality is constant.

How about the private information case? Suppose θ_t is iid and consider two different stories $\theta^t, \tilde{\theta}^t$.

$$u'(c_t(\theta^T \backslash \theta^t)) = \beta R \sum_{\theta^{t+1} \backslash \theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c_{t+1}(\theta^T \backslash \theta^t))$$

$$u'(c_t(\theta^T \backslash \tilde{\theta}^t)) = \beta R \sum_{\theta^{t+1} \backslash \theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c_{t+1}(\theta^T \backslash \tilde{\theta}^t))$$

and notice that $\pi(\theta^T \backslash \tilde{\theta}^t) = \pi(\theta^T \backslash \theta^t)$. Now suppose that $u'(c_t(\theta^T \backslash \theta^t)) > u'(c_t(\theta^T \backslash \tilde{\theta}^t))$, then there exists a history θ^{t+1} such that $\pi(\theta^{t+1} \backslash \theta^t) = \pi(\theta^{t+1} \backslash \tilde{\theta}^t)$ and:

$$u'(c_{t+1}(\theta^T \backslash \theta^t)) > u'(c_{t+1}(\theta^T \backslash \tilde{\theta}^t))$$

meaning that good shocks up to period t has persistent effect on period $t+1$ allocations. Let's talk about inequality now:

For exposition purposes, assume $u(c) = log(c)$, then $u'(c) = \frac{1}{c}$. Let's depart from the Inverse Euler Equation ($\beta R = 1$):

$$\frac{1}{u'(c_t(\theta^T))} = \mathbb{E}\left[\frac{1}{u'(c_{t+1}(\theta^T))} \backslash \theta^t\right]$$

Let's see what happens to the variance of consumption over time:

$$Var\left(\frac{1}{u'(c_t(\theta^T))}\right) = Var\left(\mathbb{E}\left[\frac{1}{u'(c_{t+1}(\theta^T))} \backslash \theta^t\right]\right)$$
$$= Var\left(\frac{1}{u'(c_{t+1}(\theta^T))}\right) - \mathbb{E}\left[Var\left(\frac{1}{u'(c_{t+1}(\theta^T))} \backslash \theta^t\right)\right]$$

If $Var\left(\frac{1}{u'(c_{t+1}(\theta^T))} \backslash \theta^t\right) > 0$ for some θ^t, then:

$$Var\left(\frac{1}{u'(c_t(\theta^T))}\right) < Var\left(\frac{1}{u'(c_{t+1}(\theta^T))}\right)$$

and therefore:

$$Var(c_t(\theta^T)) < Var(c_{t+1}(\theta^T))$$

So this is saying that inequality grows and it is efficient!! How about mobility? In short-run there is mobility. What about long-run?

Notice that $\frac{1}{u'(c_t)}$ is a martingale ($\beta R = 1$). We also know by feasibility that $\mathbb{E}\left[\frac{1}{u'(c_{t+1})}\right] < \infty$.

Martingale Convergence Theorem: If $\{x_t\}_{t=1}^{\infty}$ is a stochastic process adadpted to filteration $\{\mathcal{F}_t\}_{t=1}^{\infty}$ such that $x_t = \mathbb{E}[x_{t+1} \backslash \mathcal{F}_t]$ and $\mathbb{E}[x_t] < \infty$ for all t, then:

$$\lim_{t\to\infty} x_t \overset{a.s.}{\to} x_\infty < \infty$$

where x_∞ is a random variable with $\mathbb{E}[x_\infty] < \infty$. Therefore, $\frac{1}{u'(c_t)}$ converges to a finite number and hence there is not mobility in the long-run.

4.2 Long-run properties of efficient allocations

If you are interested in this section, read Farhi and Werning (2005, 2007, 2010), Phelan (2006) and Atkeson and Lucas (1992). We will maintain the following assumptions:

- $T = \infty$
- $Theta = \{\theta_L, \theta_H\}$
- θ_t iid over time

The proves of the following results will be loose. For more rigurousity please read the papers above.

Immiseration Result

Again, for simplicity assume $\beta R = 1$ and consider the following planning problem:

$$w_0 = max \quad \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t}{\theta_t}\right)\right] \tag{21}$$

$$s.t.$$

$$\sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t}{\theta_t}\right)\right] \geq \sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha'_t(\theta^t))) - v\left(\frac{y_t(\alpha'_t(\theta^t)}{\theta_t}\right)\right]$$

for all $\alpha' : D \to D$ (α'_t is θ^t-measurable).

$$\sum_{\theta^t} \sum_{t=1}^{T} \pi(\theta^t)[c_t(\theta^t) - y_t(\theta^t)]/R^{t-1} \leq 0$$

We can express this problem in its dual form:

$$K(w_0) = min \quad \sum_{\theta^t} \sum_{t=1}^{T} \pi(\theta^t)[c_t(\theta^t) - y_t(\theta^t)]/R^{t-1} \tag{22}$$

$$s.t.$$

$$\sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t}{\theta_t}\right)\right] \geq \sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha'_t(\theta^t))) - v\left(\frac{y_t(\alpha'_t(\theta^t)}{\theta_t}\right)\right]$$

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t}{\theta_t}\right)\right] \geq w_0$$

$K(U_0)$ would be the cost of delivering ex-ante utility U_0 for everyone. We want to write this recursively: consider an allocation $(c_t(\theta^t), y_t(\theta^t)$. Consider a history $\overline{\theta}^t$. Now, define the ex-ante utility for an agent with history $\overline{\theta}^t$ under this plan as:

$$w_t(\overline{\theta}^t) = \sum_{s=t}^{\infty} \sum_{\theta^s \setminus \overline{\theta}^t} \pi(\theta^s)\beta^{s-t} \left[u(c_s(\theta^s)) - v\left(\frac{y_s(\theta^s}{\theta_s}\right)\right]$$

Let's call $\omega_t(\overline{\theta}^t)$ the 'promised utility' after history $\overline{\theta}^t$. Denote by $(c^*_t(\theta^t), y^*_t(\theta^t))$ the solution of the problem in (22). Denote by $w^*_t(\overline{\theta}^t)$ the promised utility after history $\overline{\theta}^t$. It is possible to show that $(c^*_t(\overline{\theta}^t, \theta_{t+1}), y^*_t(\overline{\theta}^t, \theta_{t+1}), w^*_t(\overline{\theta}^t, \theta_{t+1}))$ solve the following Bellman equation at $w = w^*(\theta^t)$ (imposing $\beta R = 1$:

$$K(w) = \min_{c,y,w} \quad \sum_{\theta} \pi(\theta)[c(\theta, w) - y(\theta, w) + \beta K(w'(\theta, w))]$$

$$s.t.$$

$$u(c(\theta, w)) - v\left(\frac{y(\theta, w)}{\theta}\right) + \beta w'(\theta, w) \geq u(c(\theta', w)) - v\left(\frac{y(\theta', w)}{\theta}\right) + \beta w'(\theta', w) \quad \forall \theta, \theta'$$

$$\sum_{\theta} \pi(\theta) \left[u(c(\theta, w)) - v\left(\frac{y(\theta, w)}{\theta}\right) + \beta w'(\theta, w)\right] \geq w$$

We call this second constraint the 'promise keeping' constraint.

Proposition 11 *$K(w)$ is strictly increasing and strictly convex (assumption on $v(.)$ is needed). Also, let $\underline{w}$ and $\overline{w}$ be the lowest and highest possible values for promised utility. Then, $\lim_{w \to \underline{w}} K'(w) = 0$ and $\lim_{w \to \overline{w}} K'(w) = \lim_{w \to \overline{w}} K(w) = 0$.*

Let $\mu(\theta, \theta')$ be the multiplier on the Incentive compatibility constraint and ϕ be the multiplier on promise keeping. First order condition with respect to $c(\theta, U)$ is:

$$u'(c(\theta, w)) \left[\sum_{\theta'} \mu(\theta, \theta') - \sum_{\theta'} \mu(\theta', \theta) + \pi(\theta)\phi\right] = \pi(\theta)$$

and with respect to $w'(\theta, U)$:

$$\left[\sum_{\theta'} \mu(\theta, \theta') - \sum_{\theta'} \mu(\theta', \theta) + \pi(\theta)\phi\right] = \pi(\theta) K'(w'(\theta, w)) \tag{23}$$

and therefore:

$$K'(w'(\theta, w)) = \frac{1}{u'(c(\theta, w))}$$

this brings to the table the following lemma:

Lemma 5 *Given any $w \in [\underline{w}, \overline{w}]$, if $w'(\theta, w) = w'(\theta', w)$ for some θ, $\theta' \in \Theta$, then $c(\theta, w) = c'(\theta, w)$*

Grab equation (23) and sum over θ:

$$\sum_{\theta} \pi(\theta) K'(w'(\theta, w)) = \sum_{\theta} \sum_{\theta'} \mu(\theta, \theta') - \sum_{\theta} \sum_{\theta'} \mu(\theta', \theta) + \phi = \phi$$

From the envelope condition we have:

$$K'(w) = \phi$$

therefore

$$K'(w) = \sum_{\theta} \pi(\theta) K'(w'(\theta, w))$$

Now start from a given w_0 and construct the stochastic process w_t as follows:

$$w_{t+1} = w'(\theta_t, w_t)$$

then:

$$K'(w_t) = \mathbb{E}_t[K'(w_{t+1})]$$

hence w_t is a martingale. By martingale convergence theorem there must exist a w_∞ such that $w_t \to w_\infty$. Suppose $K'(w_\infty) > 0$. Note that convergence implies:

$$w'(\theta, w_\infty) = w'(\theta', w_\infty) \quad \forall \theta, \theta'$$

and therefore

$$c(\theta, w_\infty) = c(\theta', w_\infty) \quad \forall \theta, \theta'$$

and then incentive compatibility implies:

$$y(\theta, w_\infty) = y(\theta', w_\infty) \quad \forall \theta, \theta'$$

but we know from our two type example that the planner can do better by differentiating various θ types. Therefore, this is a contradiction. Hence $K'(w_\infty) = 0$ and $w_\infty = \underline{w}$.

4.2.1 No Immiseration Result

Consider the following planner in which the Planner valus future consumption more than the agent ($\hat{\beta} > \beta$):

$$w_0 = max \quad \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \hat{v}\left(\frac{y_t(\theta^t}{\theta_t} \right) \right] \tag{24}$$

$$s.t.$$

$$\sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - \hat{v}\left(\frac{y_t(\theta^t}{\theta_t} \right) \right] \geq \sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha'_t(\theta^t))) - \hat{v}\left(\frac{y_t(\alpha'_t(\theta^t)}{\theta_t} \right) \right]$$

for all $\alpha' : D \to D$ (α'_t is θ^t-measurable).

$$\sum_{\theta^t} \sum_{t=1}^{T} \pi(\theta^t)[c_t(\theta^t) - y_t(\theta^t)]/R^{t-1} \leq 0$$

and make the following assumptions:

- $\hat{\beta} R = 1$
- $\hat{v}\left(\frac{y}{\theta}\right) = \frac{v(y)}{\theta}$
- $u(.)$ is unbounded below, hence $\underline{w} = -\infty$
- $\mathbb{E}\left[\frac{1}{\theta}\right] = 1$

Let's assume that the problem above has a solution and let's denote by $\hat{\lambda}$ the multiplier on the resources constraint. Then the solution of the problem above must also be a solution of:

$$P(w_0) = max \quad \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] \tag{25}$$

$$s.t.$$

$$\sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} \right] \geq \sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha'_t(\theta^t))) - \frac{v(y_t(\alpha'_t(\theta^t)))}{\theta_t} \right]$$

for all $\alpha' : D \to D$ (α'_t is θ^t-measurable).

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} \right] \geq w_0$$

We can show that this problem also solves the following Bellman equation (after any history):

$$P(w) = \max_{c,y,w'} \sum_{\theta} \pi(\theta) \left[u(c(\theta, w)) - \frac{v(y(\theta, w))}{\theta} - \hat{\lambda} c(\theta, w) + \hat{\lambda} y(\theta, w) + \hat{\beta} P(w'(\theta, w)) \right] \tag{26}$$

$$s.t.$$

$$u(c(\theta,w)) - \frac{v(y(\theta,w))}{\theta} + \beta w'(\theta,w) \geq u(c(\theta',w)) - \frac{v(y(\theta',w))}{\theta} + \beta w'(\theta',w) \quad \forall \theta,\theta'$$

$$\sum_{\theta} \pi(\theta)\left[u(c(\theta,w)) - \frac{v(y(\theta,w))}{\theta} + \beta w'(\theta,w)\right] \geq w$$

what we want to show is that in this problem, in the long-run the promised utility can not be misery $w_\infty \to -\infty$. To show this, we use two lemmas:

Lemma 6 *The value function $P(w)$ is strictly concave and continuously differentiable on $(-\infty,\overline{w})$. Furthermore:*

$$\lim_{v\to-\infty} P(w) = \lim_{v\to\overline{w}} P(w) = \lim_{v\to\overline{w}} P'(w) = -\infty$$

and

$$\lim_{v\to-\infty} P'(w) = 1$$

I will skip the proof of concavity and differentiability (should be quite standard). I will sketch part of what I understood from Hosseini notes about the rest of the lemma:

$$P_{FI}(w) = max \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t)\right]$$

$$s.t.$$

$$\sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t}\right] \geq w$$

In other words, $P_{FI}(w)$ is the value to the planner from delivering utility w to individual when the IC constraint is ignored. This would naturally imply that $P_{FI}(w) > P(w)$. Also, $P_{FI}(w)$ is strictly concave and differentiable. Now consider the following maximization problem:

$$m = \max_{c,y,\theta} \quad u(c) - \frac{v(y)}{\theta} - \hat{\lambda} c + \hat{\lambda} y$$

The problem above has a solution $u'(c) = \hat{\lambda}, v'(y) = \hat{\lambda}\theta$, and θ belongs to a compact set. Now, notice that:

$$\begin{aligned}
P_{FI}(w) &= \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t)\right] \\
&= \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t)\right] \\
&+ \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t)\right] \\
&- \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t)\right]
\end{aligned}$$

$$= w + \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t)[-\hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t)] +$$

$$\sum_{t=1}^{T} (\hat{\beta}^{t-1} - \beta^{t-1}) \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right]$$

$$\leq w + \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t)[-\hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t)] + m \left(\frac{1}{1-\hat{\beta}} - \frac{1}{1-\beta} \right)$$

$$\leq w - \hat{\lambda} \tilde{K}(w) + m \left(\frac{1}{1-\hat{\beta}} - \frac{1}{1-\beta} \right)$$

in which

$$\tilde{K}(w) = min \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t)[c_t(\theta^t) - y_t(\theta^t)]$$

$$s.t.$$

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} \right] \geq w$$

Notice that $\tilde{K}(w)$ is strictly convex and differentiable and also $\lim_{w \to -\infty} \tilde{K}'(w) = 0$. Define $P_{max}(w) = w - \tilde{K}(w) + m$. Then $P_{max}(w) \geq P_{FI}(w)$ and both are strictly concave. Also, $\lim_{w \to -\infty} P_{FI}(w) \leq \lim_{w \to -\infty} P_{max}(w) = -\infty$. Therefore, $\lim_{w \to -\infty} P'_{FI}(w) \leq \lim_{w \to -\infty} P'_{max}(w) = 1$.

Notice also that $\lim_{w \to -\infty} P(w) \leq \lim_{w \to -\infty} P_{FI}(w) = -\infty$ and since both are strictly concave:

$$\lim_{w \to -\infty} P'(w) \leq \lim_{w \to -\infty} P'_{FI}(w) = 1$$

$$\lim_{w \to -\infty} P'(w) \leq 1$$

Now consider allocations $(c(w_0, \theta^t)), y(w_0, \theta^t))$ that solve the original problem. Let's say they attain the value $P(w_0)$ and define new allocations $(\tilde{c}(w, \theta^t), \tilde{y}(w, \theta^t))$ for $w \leq w_0$ as:

$$\tilde{c}(w, \theta^t) = c(w_0, \theta^t) \quad \forall \theta^t \forall t$$

$$\tilde{y}(w, \theta^t) = y(w_0, \theta^t) \quad \forall \theta^t \forall t > 1$$

$$\tilde{y}(w, \theta_1) = v^{-1}(v(y(w_0, \theta_1)) + w_0 - w)$$

Now define $P_m(w)$ for $w \leq w_0$ as:

$$P_m(w) = \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(\tilde{c}(w, \theta^t)) - \frac{v(\tilde{y}(w, \theta^t))}{\theta} - \hat{\lambda} \tilde{c}(w, \theta^t) + \hat{\lambda} \tilde{y}(w, \theta^t) \right]$$

$$= \sum_{\theta_1} \pi(\theta_1) \left[u(\tilde{c}(w_0, \theta_1)) - \frac{v(y(w_0, \theta_1)) + w_0 - w}{\theta_1} - \hat{\lambda} c(w_0, \theta_1) + \hat{\lambda} \tilde{y}(w_0, \theta_1) \right] +$$

$$\sum_{t=2}^{T}\hat{\beta}^{t-1}\sum_{\theta^T\in D}\pi(\theta^t)\left[u(\tilde{c}(w_0,\theta^t))-\frac{v(\tilde{y}(w_0,\theta^t))}{\theta}-\hat{\lambda}\tilde{c}(w_0,\theta^t)+\hat{\lambda}\tilde{y}(w_0,\theta^t)\right]$$

$$=\sum_{\theta_1}\pi(\theta_1)\left[\frac{w-w_0}{\theta_1}+\hat{\lambda}\tilde{y}(w,\theta_1)\right]+\sum_{\theta_1}\pi(\theta_1)\left[u(\tilde{c}(w_0,\theta_1))-\frac{v(y(w_0,\theta_1))}{\theta_1}-\hat{\lambda}c(w_0,\theta_1)\right]+$$

$$\sum_{t=2}^{T}\hat{\beta}^{t-1}\sum_{\theta^T\in D}\pi(\theta^t)\left[u(\tilde{c}(w_0,\theta^t))-\frac{v(\tilde{y}(w_0,\theta^t))}{\theta}-\hat{\lambda}\tilde{c}(w_0,\theta^t)+\hat{\lambda}\tilde{y}(w_0,\theta^t)\right]$$

Note that $P_m(w)$ is strictly concave, $P_m(w)\leq P(w)$. Also, only the first term depends on w. Therefore:

$$P_m'(w)=1-\hat{\lambda}\mathbb{E}\left[\frac{1}{v'(v(y(w,\theta_0))+w_0-w)}\right]$$

and $\lim_{w\to-\infty}P_m'(w)=1$.

Therefore, $\lim_{w\to-\infty}P'(w)\geq lim_{w\to-\infty}P_m'(w)=1$. Hence, it has been proved that $\lim_{w\to-\infty}P'(w)=1$.

Q.E.D.

In the next lemma we will be showing that $1-P(w'(w,\theta))$ can be bounded above and below for all θ. Again, let's rewrite problem (23) as:

$$P(w)=\max_{c,y,w'}\sum_{\theta}\pi(\theta)\left[u(c(\theta,w))-\frac{v(y(\theta,w))}{\theta}-\hat{\lambda}c(\theta,w)+\hat{\lambda}y(\theta,w)+\hat{\beta}P(w'(\theta,w))\right]\tag{27}$$

$$s.t.$$

$$u(c(\theta,w))-\frac{v(y(\theta,w))}{\theta}+\beta w'(\theta,w)\geq u(c(\theta',w))-\frac{v(y(\theta',w))}{\theta}+\beta w'(\theta',w)\quad\forall\theta,\theta'$$

$$\sum_{\theta}\pi(\theta)\left[u(c(\theta,w))-\frac{v(y(\theta,w))}{\theta}+\beta w'(\theta,w)\right]\geq w$$

Suppose only the IC constraints of the high types are binding. Denote by μ and ϕ the multipliers on IC and promise-keeping, respectively. Taking FOCs we have: