Introduction to Heterogeneous Agent Models

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RIEF

Introduction

- Heterogeneity can be introduced in NGM in several ways:
  1. Preferences
  2. Initial conditions
  3. Productivity
- We can introduce heterogeneity both on consumers and firms.
- Heterogeneous agent model allow us to tackle new questions regarding, for example, inequality.
- They also allow to answer better old questions regarding business cycles and economic growth.

A Model with Incomplete Markets and Idiosyncratic Shocks

The following seminal papers are the origin of this literature:

- Bewley (JET-1977).
- Aiyagari (QJE-1994, JPE-1995).
- Hugget (JEDC-1993, JME-1996).

The Basic Model - Idiosyncratic Productivity

- Continuum of agents (measure 1).
- In each period, agents draw λ_t^i from a Markovian distribution. λ_t^i determines idiosyncratic productivity.
- Ex-ante agents are identical (same initial assets a_0^i and initial productivity λ_0^i.
- Their different stories of λ_t^i will make them different in the future.
- In particular, agents will have different assets a_t^i and productivity λ_t^i

The Basic Model - Aggregate uncertainty?

- Therefore, in period 0 everyone solves the same problem.
- By LLN, there is individual uncertainty but NOT aggregate uncertainty:

$$L_t \equiv \int_0^1 \lambda_t^i di = 1$$

The Basic Model - Incomplete Markets

- There is only one available asset in the economy. This asset is assumed to be risk-free.
- Agents only can insure themselves against negative shocks by accumulating this asset.
- A borrowing-constraint is also assumed to be present.

The Basic Model - Individual Problem

- Each agent solves:

$$\underset{c_t, a_{t+1}}{Max}\, \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.$$

$$c_t + a_{t+1} = R_t a_t + w_t \lambda_t$$

$$a_{t+1} \geq -\phi$$

$$a_0, \lambda_0 \quad given$$

Competitive Equilibrium

A CE in this economy is a set of contingent plans for individual quantities $c_t(\lambda^t)$, $a_{t+1}(\lambda^t)$, sequences for aggregate quantities Y_t, K_t and prices w_t R_t such that:

- Given $a_0 \geq -\phi$, $\lambda_0 > 0$, w_t, R_t and the stochastic process λ, $c_t(\lambda^t)$ and $a_{t+1}(\lambda^t)$ solves:

$$\underset{c_t, a_{t+1}}{Max} \sum_{t=0}^{\infty} \sum_{\lambda^t \in \Lambda^t} \beta^t \pi(\lambda^t) u(c_t(\lambda^t)) \quad s.t.$$

$$c_t(\lambda^t) + a_{t+1}(\lambda^t) = w_t \lambda_t + R_t a_t(\lambda^{t-1})$$

$$a_{t+1}(\lambda^t) \geq -\phi \quad \forall \lambda^t, \forall t$$

Competitive Equilibrium

- In each period, given w_t and R_t, Y_t and K_t solve the firm's problem:

$$Max \quad Y_t - w_t - [R_t - (1-\delta)]K_t$$
$$Y_t = f(K_t)$$

- In each period, markets clear:

$$Y_t = \sum_{\lambda^t} \pi(\lambda^t)\left[c_t(\lambda^t) + a_{t+1}(\lambda^t) - (1-\delta)a_t(\lambda^{t-1})\right]$$

$$K_t = \sum_{\lambda^t} \pi(\lambda^t) a_t(\lambda^{t-1})$$

Some Remarks

- Agents make exactly the same contingent plans. Why?
- Consumption and assets of each individual depends on the historiy of their shocks.
- By LLN, $\pi(\lambda^t)$ is the fraction of agents with a story λ^t.

Complete Markets and Arrow-Debreu

- Efficient allocation in this economy solves:

$$\max_{\substack{K_{t+1}\\C_t}} \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$s.t.$$

$$C_t + K_{t+1} - (1-\delta)K_t = f(K_t) \quad \forall t$$

$$K_0 \quad given$$

- Since ex-ante everyone is identical $C_t = c_t$.
- That means, efficient allocaiton has perfect insurance!

Is it possible to implement the efficient allocation?

- With incomplete markets and borrowing constraints, it is not.
- We could decentralize the solution in two different ways though:
  1. With Arrow-Debreu securities:

$$\sum_{t=0}^{\infty} \sum_{\lambda^t \in \Lambda^t} p_t(\lambda^t) c_t(\lambda^t) = \sum_{t=0}^{\infty} \sum_{\lambda^t \in \Lambda^t} p_t(\lambda^t) w_t \lambda_t$$

  2. With Sequential Markets

$$c_t(\lambda^t) + \sum_{\lambda^{t+1} \backslash \lambda^t} a_{t+1}(\lambda^{t+1} \backslash \lambda^t) = w_t \lambda_t + R_t a_t(\lambda^t \backslash \lambda^{t-1})$$

Recursive Formulation

- Let's go back to our incomplete markets model.
- Individual state variables are a and λ.
- Fraction of agents with assets $a \leq a^*$ and productivity $\lambda \leq \lambda^*$ is $\mu(a^*, \lambda^*)$:

$$\mu : S \equiv [-\phi, \infty)x[\lambda_{min}, \lambda_{max}] \rightarrow [0, 1]$$

$$\lim_{a \rightarrow \infty} \mu_t(a, \lambda_{max}) = 1$$

- μ is the aggregate state variable:

$$\int_S a d\mu(a, \lambda) = K \quad \int_S \lambda d\mu(a, \lambda) = L = 1$$

Recursive Competitive Equilibrium

A Recursive Competitive Equilibrium is a set of functions $v(a, \lambda, \mu)$, $c(a, \lambda, \mu)$, $a'(a, \lambda, \mu)$, prices $w(\mu)$ and $R(\mu)$, aggregate capital $K(\mu)$ and a law of motion $\Gamma(\mu)$ such that:

- For each triple (a, λ, μ), given the functions w, r, Γ, the value function $v(a, \lambda, \mu)$ solves the following Bellman equation:

$$
\begin{aligned}
v(a, \lambda, \mu) &= \underset{c, a'}{Max}\, u(c) + \beta \mathbb{E}_{\lambda} v(a', \lambda', \mu') \\
s.t. \quad c + a' &= w(\mu)\lambda + R(\mu)a \\
a' &\geq -\phi \\
\lambda' &\sim \Pi(\lambda) \\
\mu' &= \Gamma(\mu)
\end{aligned}
$$

$c(a, \lambda, \mu)$, $a'(a, \lambda, \mu)$ are optimal decision rules for this problem.

Recursive Competitive Equilibrium

- For each distribution prices satisfy:

$$R(\mu) = f'(K(\mu)) + (1 - \delta)$$

$$w(\mu) = f(K(\mu)) - f'(K(\mu))K(\mu)$$

- For each distribution μ, markets clear:

$$f(K(\mu)) = \int_S [c(a, \lambda, \mu) + a'(a, \lambda, \mu) - (1 - \delta)a] d\mu(a, \lambda)$$

$$K(\mu) = \int_S a d\mu(a, \lambda) \qquad 1 = \int_S \lambda d\mu(a, \lambda)$$

- For each μ, Γ is consistent with individual decisions.

Steady State

- It is an equilibrium in which aggregate quantities C_t, K_t and prices w_t and R_t are constant.
- This means, it is an equilibrium in which $\mu* = \Gamma(\mu*)$

Interest rate and incomplete markets

- Our analysis is focused in steady state.
- In steady state, the interest rate with incomplete markets will be lower than in a complete markets scenario.
- Why? Precautionary savings!
- Let's show this result formally.

Interest rate and incomplete markets

- In steady state, the Bellman equation is:

$$v(a,\lambda) = \underset{c,a'}{Max}\, u(c) + \beta \mathbb{E}_\lambda v(a',\lambda')$$

$$s.t.$$

$$c + a' = w^* \lambda + R^* a$$

$$a' \geq -\phi$$

Taking FOC we have:

$$-u'(c) + \beta \mathbb{E}_\lambda v_a(a',\lambda') \leq 0 \quad (= \text{ if } a > -\phi)$$

combining this with Benveniste-Scheinkman we have:

$$u'(c) \geq \beta R^* \mathbb{E}_\lambda u'(c) \quad (= \text{ if } a > -\phi)$$

Result: In any steady state for this economy: $R < \frac{1}{\beta}$

- I show this result for an $i.i.d$ shock λ.
- Define total resources as $z \equiv w^*\lambda + R^*a + \phi$ and rewrite problem as:

$$v(z) = \max_{c,a'} \left\{ u(c) + \beta \mathbb{E} v(z') \right\}$$

$$c + a' = z - \phi$$

$$a' \geq -\phi$$

$$z' = w^*\lambda' + R^*a' + \phi$$

labor and capital income are perfect substitutes. Agent only cares about the summation of those.

Result: In any steady state for this economy: $R < \frac{1}{\beta}$

- Using properties of T-operator (Bellman), show $v(z)$ is strictly concave.
- Apply Benveniste-Scheinkman and get:

$$v'(z) = R^* u'(c(z))$$

Since u and v are strictly concave in z, c is strictly increasing in z.

- Assets have an upper bound if there is $\bar{z}$ such that:

$$\bar{z} = w^* \lambda_{max} + R^* a'(\bar{z}) + \phi$$

Result: In any steady state for this economy: $R < \frac{1}{\beta}$

- Write Euler Equation as:

$$v'(z) \geq \beta R^* \mathbb{E} v'(z')$$

Then

$$v'(\bar{z}) \geq \beta R^* \mathbb{E} v'(w^* \lambda' + R^* a'(\bar{z}) + \phi)$$

- Since v' is strictly decreasing:

$$\mathbb{E} v'(w^* \lambda' + R^* a'(\bar{z}) + \phi) = \sum_{\lambda} \pi(\lambda) v'(w^* \lambda + R^* a'(\bar{z}) + \phi)$$

$$> v'(w^* \lambda_{max} + R^* a'(\bar{z}) + \phi) = v'(\bar{z})$$

Result: In any steady state for this economy: $R < \frac{1}{\beta}$

- Combining the last 2 steps we have:

$$v'(\bar{z}) > \beta R^* v'(\bar{z})$$

  then $R^* \geq \frac{1}{\beta}$ implies a contradiction $(v'(\bar{z}) > v'(\bar{z}))$
- Concluding $R \geq \frac{1}{\beta}$ implies there is not a superior $\bar{z}$, which means assets grow without limit.
- Then, in any steady state we should have $R^* < \frac{1}{\beta}$.

Precautionary Savings

- Going back to the planner's problem that represents the complete-markets solution:

$$Max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.$$

$$C_t + K_{t+1} - (1-\delta)K_t = f(K_t)$$

$$K_0 \quad given$$

Taking FOC we will obtain:

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = f'(K_{t+1}) + (1-\delta) = R_t$$

Precautionary Savings

- Then in a steady-state equilibrium:

$$R^* = f'(K^*) + (1-\delta) = \frac{1}{\beta}$$

This means:

$$R^*_{eq} < R^*_{plan} = \frac{1}{\beta}$$

which implies

$$K^*_{eq} > K^*_{plan}$$

What do we learn from this?

- In a model with idiosyncratic shocks, incomplete markets and credit constraints, agents save more than the efficient amount.
- This is because they need to mitigate the effects of bad realizations of productivity shocks.
- When agents are hit by negative shock, agents may be affected by the borrowing constraints, so they need to use their own savings.
- Therefore, agents save for **precaution**.