Introduction to Measure Theory

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RIEF

Introduction

- Most of the results in Chapter 4 carry over almost without change to situations in which the return function is subject to stochastic shocks and the objective is to maximize the expected value of discounted returns.
- To show this, it is convenient to draw upon some of the terminology and results from modern probability theory and from the theory of Markov Processes.
- In particular, we will be interested, for instance, in solving problems of the form:

$$v(x,z) = max_{0 \leq y \leq f(x)z}\{U(f(x)z - y) + \beta \mathbb{E}[v(y,z')]\} \qquad (1)$$

Sigma Algebra and Measurable Spaces

- **Def:** Let S be a set and let $\mathbb{L}$ be a family of subsets of S. $\mathbb{L}$ is called a $\sigma - algebra$ if:
  1. $\Phi, S \in \mathbb{L}$
  2. $A \in \mathbb{L}$ implies $A^c = S \backslash A \in \mathbb{L}$; and
  3. $A_n \in \mathbb{L}$, $n = 1, 2, ...,$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathbb{L}$.

  For a set S what we want to ask is in what family of subsets of S are measures, including probability measures.

- **Def:** A pair $(S, \mathbb{L})$ where S is a set and $\mathbb{L}$ is a $\sigma - algebra$ of its subsets is called a **measurable space**. Any set $A \in \mathbb{L}$ is called a $\mathbb{L}$-measurable set.

Borel Algebra

- An important example of σ-algebra is the one generated by the collection of all open intervals.
- In particular, define $\mathfrak{A}$ as the collection of all open intervals in $\mathbb{R}$. That is, all the intervals of the form $(-\infty, b)$, (a, b), $(a, +\infty)$ and $(-\infty, +\infty)$. Note that every σ- algebra containing $\mathfrak{A}$ must also contain all of the closed intervals.
- The smallest σ-algebra containing all of the open sets is a class that is used in many applications and it is called the Borel's σ-algebra.

- **Measures:** Let $(S, \mathbb{L})$ be a measurable space. A **measure** is an extended-real value function $\mu : \mathbb{L} \to \overline{\mathbb{R}}$ such that:
  1. $\mu(\Phi) = 0$;
  2. $\mu(A) \geq 0$, all $A \in \mathbb{L}$
  3. If $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence of subsets in $\mathbb{L}$, then $\mu(\{A_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \mu(A_n)$.

  Then a measure is non-negative, assigns zero to the null set and is countably additive. If $\mu(S) < \infty$, then μ is finite.

Measure Space

Measure Space: is a triple $(S, \mathbb{L}, \mu)$ where S is a set, $\mathbb{L}$ is a σ-algebra of its subsets, and μ is a measure defined on $\mathbb{L}$.

- Given a measure space, we say that a proposition holds *mu*-almost everywhere if $\exists$ a set $A \in \mathbb{L}$ with $\mu(A) = 0$ such that the proposition holds on A^c
- If $\mu(S) = 1$, then μ is a probability measure and $(S, \mathbb{L}, \mu)$ is called a probability space.

Completition

Let $(S, \mathbb{L}, \mu)$ be a measure space. Let $A \in \mathbb{L}$ be any set with measure zero, and let C be any subset of A. Let $\mathcal{C}$ be the family of all such sets. That is:

$$\mathcal{C} = \{C \subset \mathbb{L} : C \subseteq A \text{ for some A} \in \mathbb{L} \text{ with } \mu(A) = 0\}$$

Now consider starting with any set $B \in \mathbb{L}$; and then adding and substracting from it sets in $\mathcal{C}$. The **completition** of $\mathbb{L}$ is the family $\mathbb{L}'$ of sets constructed in this way. That is:

$$\mathbb{L}' = \{B' \subseteq S : B' = (B \cup C_1 \backslash C_2, B \in \mathbb{L}, C_1, C_2 \in \mathcal{C}\}$$

For any Euclidean space $\mathbb{R}$, the completition of the Borel sets is a family called the **Lebesgue measurable sets**, and the extension to this family of the measure corresponding to length, area, and so on is called **Lebesgue measure**. When restricted to the Borel sets it is called either Lebesgue measure or **Borel measure**.

Caratheodory Extension Theorem

Theorem

Let S be a set, $\mathfrak{A}$ an algebra of its subsets, and μ a measure on $\mathfrak{A}$. Let $\mathfrak{C}$ be the completition of the smallest σ-algebra containing A. Then $\exists$ a measure μ^* on $\mathfrak{C}$, such that $\mu(\mathfrak{A}) = \mu^*(\mathfrak{A})$, all $A \in \mathfrak{A}$.

Measurable Functions

- We are interested in defining probability measures on the state space S so that we could talk sensibly about expressions like $\mathbb{E}(f)$, the expected value of a real function f(.) defined on S.

- We need to know then for which functions can expressions like $\mathbb{E}(f)$ be reasonably interpreted.

- **Definition:** Given a measurable space $(S, \mathbb{L})$, a real-valued function $f : S \to \mathbb{R}$ is **measurable with respect to** $\mathbb{L}$ (or **L-measurable**) if:

$$\{s \in S : f(s) \leq a\} \in \mathbb{L}, \quad \forall a \in \mathbb{R}$$

  If the σ-algebra is understood, such a function is called **measurable**. If the space in question is a probability space, f is called a **random variable**.

Pointwise Convergence

Theorem

Let $(S, \mathbb{L})$ *be a measurable space, and let* $\{f_n\}$ *be sequence of* $\mathbb{L}$*-measurable functions converging pointwise to* f*:*

$$\operatorname{lím}_{n \to \infty} f_n(s) = f(s), \quad \forall s \in S$$

Then f *is also* $\mathbb{L}$*-measurable.*

Measurable Functions

- **Definition:** Let $(S, \mathbb{L})$ and $(T, \mathbb{T})$ be measurable spaces. Then the function $f : S \rightarrow T$ is **measurable** if the inverse image of every measurable set is measurable, that is, if $\{s \in S : f(s) \in A\} \in \mathbb{L}$, $\forall$ $A \in \mathbb{T}$.

- **Definition:** Let $(S, \mathbb{L})$ and $(T, \mathbb{T})$ be measurable spaces, and let Γ be a correspondence of S into T. Then the function $h : S \rightarrow T$ is a **measurable selection from** Γ if h is measurable and $h(s) \in \Gamma(s)$, $\forall s \in S$.

Measurable Selection Theorem

Theorem

Let $S \subseteq \mathbb{R}^l$ and $T \subseteq \mathbb{R}^m$ be Borel sets, with their Borel subsets $\mathbb{L}$ and $\mathbb{T}$. Let $\Gamma : S \to T$ be a nonempty compact-valued and uhc. correspondance. Then there exists a measurable selection from Γ.

Integration

- So far we have seen how to define a probability space $(Z, \mathbb{L}, \mu)$ for the random shocks, and probably you have guessed that the function v has to be $\mathbb{L}$-measurable.
- In this section we combine these to pieces to develop an integration theory.
- The integral developed here is called "Lebesgue Integral.ªnd it uses the "Lebesgue measure". This integral is more general than the Riemann integral and it includes, as well as operations like $\sum_i \pi_i f(s_i)$ involving discrete probabilities, as special cases.
- Of course, the Riemann and the Lebesgue integral coincide with each other when the former exists.
- Lebesgue's theory of integration can be extended to real-valued functions on any measure space $(S, \mathbb{L}, \mu)$. For example, if μ is a probability measure, $\int_S f(s)\mu(ds)$ is the expected value of the rv f wrt the distribution μ.